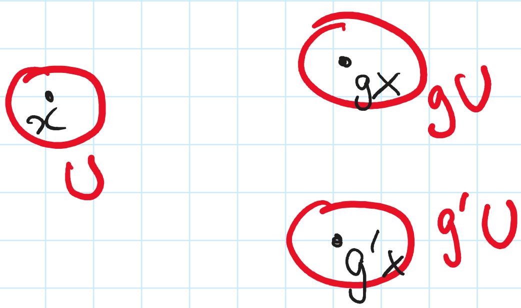
X = top. space $\circlearrowleft G$ = group

Action: for each $g \in G$, homeomorphism $g : X \to X$

$$g_1(g_2(x)) = (g_1 g_2)(x)$$

<u>Def</u> The action is <u>properly discontinuous</u>, if for all $x \in X$ there is a neighborhood U such that $U \cap gU = \emptyset$ if $g \neq e$

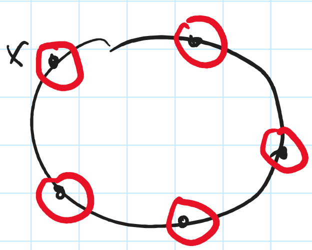


<u>Exercise</u> This implies $gU \cap g'U = \emptyset$ if $g \neq g'$

<u>Clear</u>: if action is properly discontinuous, it is free, that is, all stabilizers are trivial.

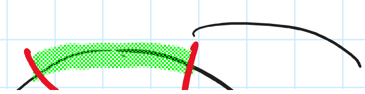
<u>Proof</u>: $x \neq gx$ if $g \neq e$

<u>Examples</u>: ① $\mathbb{Z}_n \circlearrowleft S^1$ by rotations by $\frac{2\pi}{n} \cdot k$

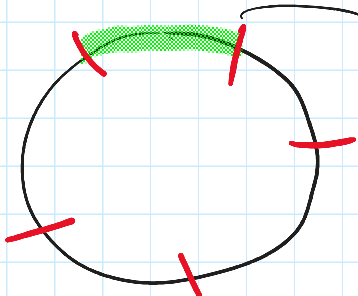


We'll be interested in a quotient $X/G = X/_{\substack{x \sim gx \\ g \in G}}$.

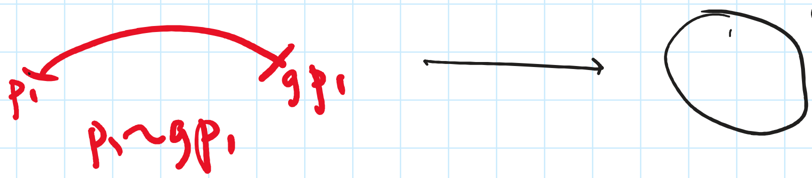
Points in X/G = orbits in X

$S^1/$  divide S^1 into n equal

$S^1/\mathbb{Z}_n$



divide S^1 into n equal intervals, each orbit intersects interval in one pt but endpoints are equivalent



$S^1 = S^1/\mathbb{Z}_n$.

Projection map: $S^1 \longrightarrow S^1/\mathbb{Z}_n = S^1$

$x \longrightarrow [x]$

it is a degree n covering of S^1.

(2) $\mathbb{Z}_2 \circlearrowright S^n$ by ± 1

$$(x_1, \ldots, x_n) \sim (-x_1, -x_2, \ldots, -x_n)$$

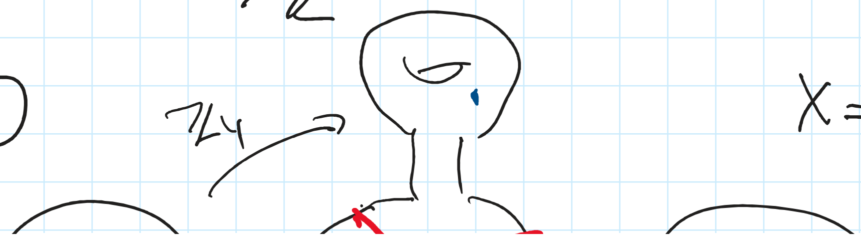
$S^n/\mathbb{Z}_2 \simeq \mathbb{R}P^n$ Each orbit has 2 points $\Rightarrow$ action is free, properly discontinuous.

(3) $\mathbb{Z}^2 \circlearrowright \mathbb{R}^2$ by translations

$$(a, b) \cdot (x_1, x_2) = (x_1 + a, x_2 + b)$$

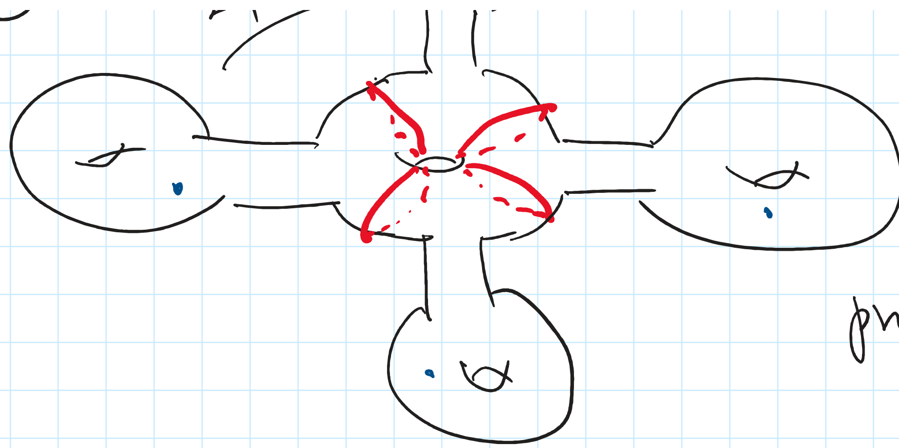
$\mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 = T^2$

(4)



$X = \Sigma_5 =$ genus 5 surface

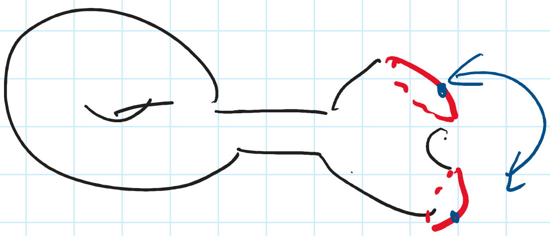
surface



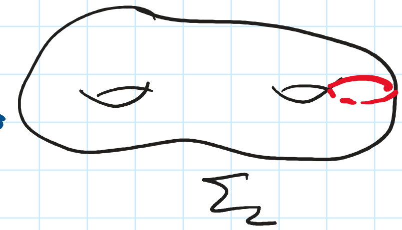
$\mathbb{Z}_4 \curvearrowright \Sigma_5$

freely

properly discontinuously



need to identify two red boundaries $\longrightarrow$



Σ_2

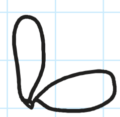
$$\Sigma_5 / \mathbb{Z}_4 \cong \Sigma_2$$

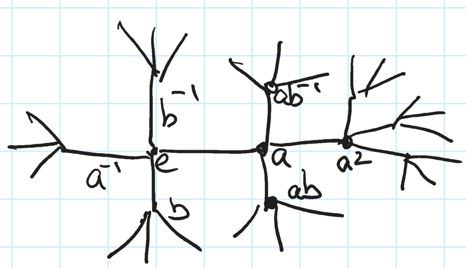
__HW#1:__ $g_1 - 1$ divides $g_2 - 1$, construct a covering map from $\Sigma_{g_2} \longrightarrow \Sigma_{g_1}$

__Hint:__ construct an action of $\mathbb{Z}_k$ on Σ_{g_2} if

$$k = \frac{g_2 - 1}{g_1 - 1}.$$

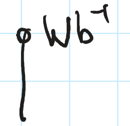
In the above example, $g_1 = 2$, $g_2 = 5$, $\frac{5-1}{2-1} = 4$.

(5) X = universal cover of 



X = infinite 4-valent tree

vertices $\longleftrightarrow$ elements in the free group F_2 with 2 generators

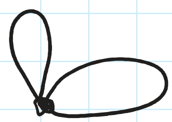


<u>Claim:</u> F_2 acts on this tree X properly & discontinuous.

$$\overset{\in}{g} \cdot \underbrace{w}_{\text{vertex}} = gw$$

Bijection on vertices, edges $\to$ edges $\Rightarrow$ continuous $X \to X$

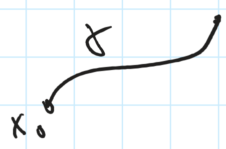
- All vertices are in one orbit
- Two types of edges, labeled by a and b

$\Rightarrow X/F_2 =$ 

$$X \longrightarrow X/F_2$$

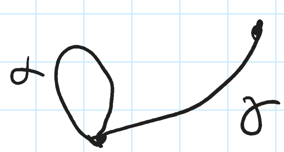
(6) $X =$ path connected, locally path conn., locally simply connected

$\tilde{X} =$ universal cover $= \{ [\gamma] : \gamma$ path on X starting at $x_0 \}$



<u>Claim!</u> $\pi_1(X)$ acts on $\tilde{X}$, properly and discontinuously and $\tilde{X}/\pi_1(X) = X$

<u>Idea:</u> $\alpha \in \pi_1(X) \hookrightarrow$ loop on X

$\alpha(\gamma) =$ 

Similarly, any subgroup $H \subset \pi_1(X)$ acts on $\tilde{X}$

and $\tilde{X}/H = X_H$ = covering space of X with $\pi_1(X_H) = H$.

(7) $SU(2) \simeq S^3$ (prove it!)

G = any finite subgroup of $SU(2)$ ← related to symmetries of regular polyhedra in $\mathbb{R}^3$

$G \curvearrowright SU(2)$ by left multiplication

$SU(2)/G = S^3/G$ = interesting 3-manifolds.

And we get a covering $S^3 \longrightarrow S^3/G$

$\pi_1(S^3) = \{e\} \Rightarrow \pi_1(S^3/G) = G$

Thm 1 Assume G acts on X properly & discontinuously. Then:

(a) $p: X \longrightarrow X/G$ is a covering map.

(b) $p_* \pi_1(X)$ is a <u>normal</u> subgroup in $\pi_1(X/G)$

$p_* : \pi_1(X) \longrightarrow \pi_1(X/G)$ group homomorphism, injective

(c) $\dfrac{\pi_1(X/G)}{p_* \pi_1(X)} = G$ if X is connected.

quotient is a group since $p_*\pi_1(X)$ is normal

Reminder: we had a bijection

$$\{\text{preimages of } x_0 \text{ under } p\} \longleftrightarrow \left\{\text{cosets } \frac{\pi_1(X/G)}{p_*\pi_1(X)}\right\}$$

$x_0 \in X/G$

Thm 2 Conversely, assume $p: \tilde{X} \longrightarrow X$ is a covering, $\tilde{X}$ connected and $p_*\pi_1(\tilde{X})$ is normal in $\pi_1(X)$

Then we can define a group $G = \dfrac{\pi_1(X)}{p_*\pi_1(\tilde{X})}$

and (a) There exists a properly discontinuous action of G on $\tilde{X}$ by "deck transformations"

(b) $\tilde{X}/G = X$

Remarks: 1) If $\pi_1(\tilde{X}) = \{e\}$ then $p_*\pi_1(\tilde{X}) = \{e\}$

Trivial group is always a normal subgroup $\Rightarrow$ Thm 2 applies and $G = \dfrac{\pi_1(X)}{\{e\}} = \pi_1(X)$ acts on $\tilde{X}$.

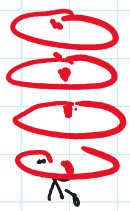
2) If $\pi_1(X)$ is abelian, then any subgroup is normal $\Rightarrow$ for any subgroup $H \subset \pi_1(X)$ we can build a cover $X_H \longrightarrow X$ with $\pi_1(X_H) = H$ and $G = \dfrac{\pi_1(X)}{H}$ acts on X_H properly & discontinuously.

Deck transformations

$p : \tilde{X} \to X$ covering.

$$\begin{tikzcd} \tilde{X} \arrow[rr, "g"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

$g : \tilde{X} \to \tilde{X}$ homeomorphism which commutes with projection to X, that is, $p \circ g = p$



$\tilde{X}$

$g(\tilde{x}_0)$ = another preimage of x_0 under p

X

$p(g(\tilde{x}_0)) = p(\tilde{x}_0) = x_0$

Roughly speaking, deck transformations = permutations of the fibers of the covering map.

Exercise $p : X \longrightarrow X/G$ as above

Then the group of deck transformations $= G$.