Lecture 21 (3/6)

Monday, March 6, 2023 10:00 AM

Review session: 215A + 239

3/20 + 3/22

Algebra Mon 1:30 – 3:30

Wed Analysis 3:30 – 4:30

Final: Thursday 3/23

Prelim: Tuesday 3/28

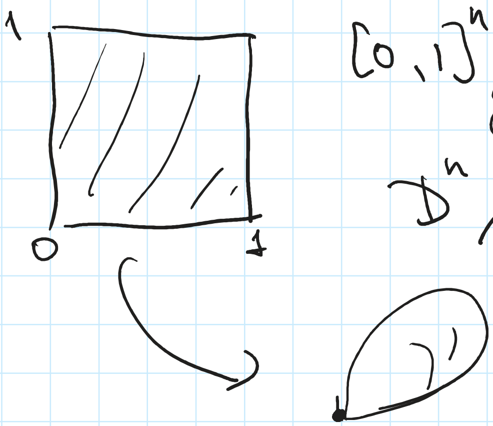
Higher homotopy groups

$$\pi_n(X) = \left\{ \begin{array}{l} \text{maps } (S^n, s_0) \xrightarrow{f} (X, x_0) \\ f(s_0) = x_0 \end{array} \right\} \Big/ \text{up to homotopy}$$

n-dim. cube

$$= \left\{ \begin{array}{l} \text{maps } [0,1]^n \xrightarrow{f} X \\ \text{such that } f(\partial [0,1]^n) = x_0 \end{array} \right\} \Big/ \text{up to homotopy}.$$

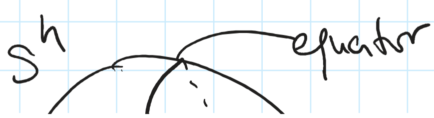
$\underline{\underline{n=2}}$



$$[0,1]^n / \partial([0,1]^n) = S^n$$
$$\cong$$
$$D^n / \partial D^n$$

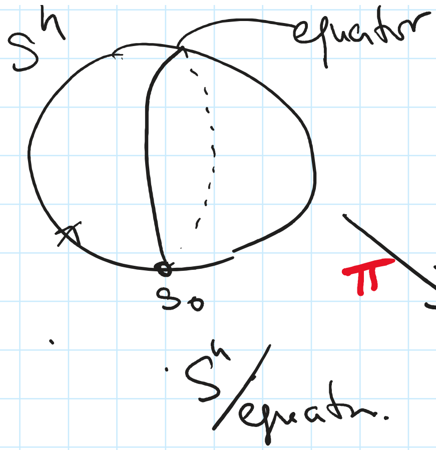
Clearly, for $n=1$ we recover the definition of π_1.

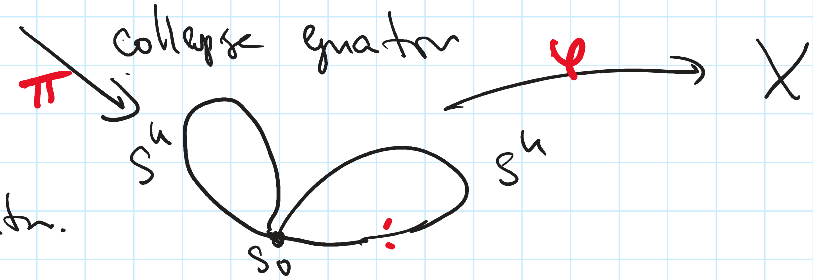
<u>Thm</u> $\pi_n(X, x_0)$ is a group with the following multiplication:



$f, g : S^n \longrightarrow X$

$$f, g : S^n \longrightarrow X$$
$$s_0 \longrightarrow x_0$$

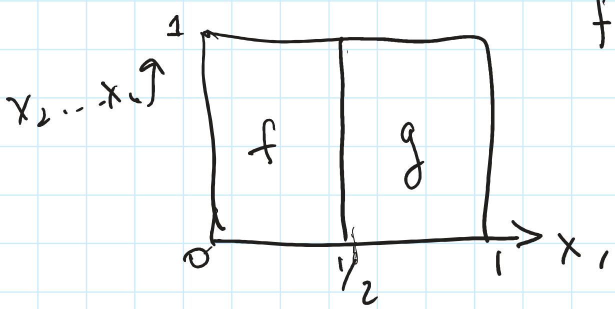




$$\varphi = \begin{cases} f & \text{on left sphere} \\ g & \text{on right sphere.} \end{cases}$$

$$f \cdot g = \varphi \circ \pi .$$

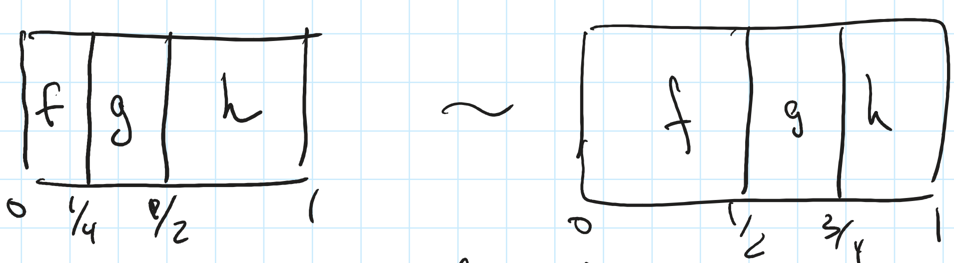
<u>Alternatively</u>:



$$f \cdot g\,(x_1, \dots, x_n) = \begin{cases} f(2x_1, x_2, \dots, x_n), & 0 \le x_1 \le \frac{1}{2} \\ g(2x_1 - 1, x_2, \dots, x_n) & \frac{1}{2} \le x_1 \le 1 \end{cases}$$

Easy to check:

- glues to a continuous map $f \cdot g : [0,1]^n \longrightarrow X$
- all the boundary is mapped to $x_0 \in X$
- Composition is associative.



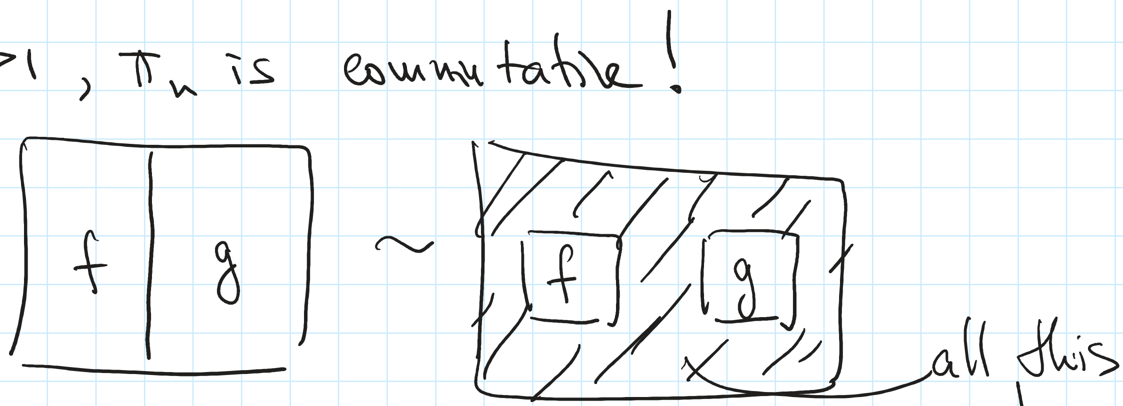
- Same definition as for S^n + equator.

- Inverse: $\bar{f}(x_1, \dots, x_n) = f(1-x_1, x_2, \dots, x_n) : [0,1]^n \to X$
- Identity = constant map, $f\bar{f} \sim \bar{f}f \sim e$.

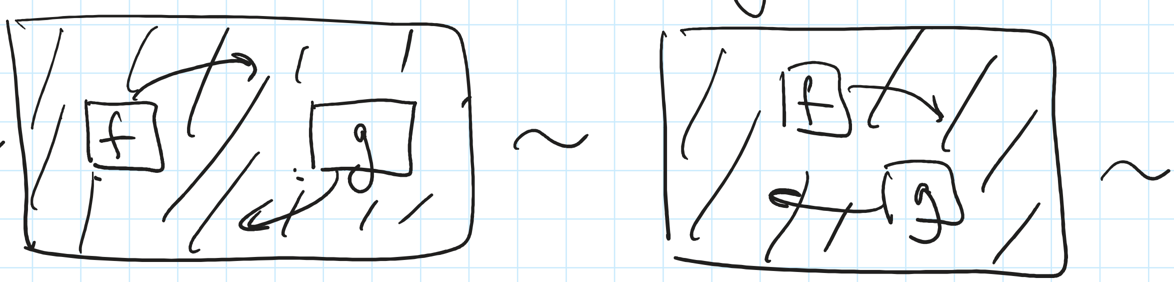
All very similar to π_1.

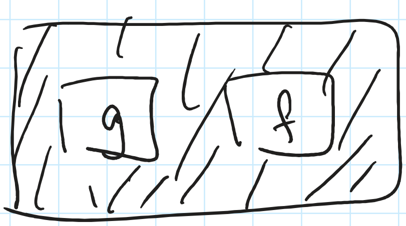
Thm For $n > 1$, π_n is commutative!

Proof

$f \circ g =$ 

We can move the boxes containing f and g continuously around:

$f \circ g \sim$ 



$\sim g \circ f$. ∎

Thm (a) $\pi_n(S^k) = 0$ if $k > n$

(b) $\pi_n(S^n) = \mathbb{Z}$ ← proof on Wednesday

Proof of (a): $S^n \xrightarrow{f} S^k$, $n < k$ use Cellular Approximation Theorem:

f is homotopic to a cellular map, so that
$\tilde{f}(S^n) \subset n$-skeleton of S^k

Choose cell decomposition of S^k with one 0-cell
one k-cell

$\Rightarrow \tilde{f}(S^n) = 0$-cell $= n$-skeleton of S^k.

<u>Fact</u> $\pi_n(S^k)$ could be very complicated if $k<n$!

In particular, $\pi_3(S^2) = \mathbb{Z} \leftarrow$ will prove this next week.

<u>Hopf fibration</u>: $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2_{z_1, z_2}$

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

$$f: S^3 \longrightarrow S^2 = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$$

$$(z_1, z_2) \longrightarrow [z_1 : z_2] = \text{complex line through } (z_1, z_2) \text{ in } \mathbb{C}^2$$

$$f^{-1}([z_1 : z_2]) = \{\text{all points in } S^3 \cap \text{complex line } [z_1 : z_2]\}$$

$$= \{(\lambda z_1, \lambda z_2), \lambda \neq 0\} \subset S^3$$

$$|\lambda z_1|^2 + |\lambda z_2|^2 = |\lambda|^2 \underbrace{(|z_1|^2 + |z_2|^2)}_{1} = 1$$

$$\Leftrightarrow |\lambda| = 1$$

Therefore, $\boxed{f^{-1}([z_1 : z_2]) = S^1}$

<u>Claim:</u> This represents a map $f: S^3 \to S^2$, not homotopic

to a constant map $\Rightarrow$ nontrivial class in $\pi_3(S^2)$

In fact, $\pi_3(S^2) \cong \mathbb{Z}$ and this is the generator.

Very hard open problem in topology: compute $\pi_n(S^k)$ in general.

<u>Thm</u> $p: \tilde{X} \longrightarrow X$ covering map, then

$$\pi_n(\tilde{X}) = \pi_n(X) \quad \text{for} \quad n > 1$$

<u>Proof</u>:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ S^n \arrow[r, "f"] \arrow[ur, dashed] & X \end{tikzcd}$$

Last week: if $f_* \pi_1(S^n) \subset p_* \pi_1(\tilde{X})$ then f lifts to a map $\tilde{f}: S^n \to \tilde{X}$.

Since $\pi_1(S^n) = \{e\}$ for $n > 1$, we get that $f_* \pi_1(S^n) = \{e\} \subset p_* \pi_1(\tilde{X})$ for any f, and we get a bijection $\{f: S^n \longrightarrow X\}$ and $\{\tilde{f}: S^n \longrightarrow \tilde{X}\}$.

<u>Ex</u> $\underbrace{\pi_n(S^1) = \pi_n(\mathbb{R})}_{\text{universal cover}} = \pi_n(pt) = 0$

$n > 1$

$$\pi_n(T^2) = \pi_n(\mathbb{R}^2) = \pi_n(pt) = 0$$

$\pi_n(\mathbb{R}P^n) = \pi_n(S^n) = \mathbb{Z} \longleftarrow$ generated by the projection $S^n \to \mathbb{R}P^n$.

(same n!) cover $S^n \longrightarrow \mathbb{R}P^n$

$\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$ $\quad$ $\underset{\underset{\pi_2(S^2)}{\shortparallel}}{\pi_2(\mathbb{R}P^2)} = \mathbb{Z}$ $\quad$ $\underset{\underset{\pi_3(S^2)}{\shortparallel}}{\pi_3(\mathbb{R}P^2)} = \mathbb{Z}$

$$\pi_2(S^2) = \mathbb{Z}, \qquad \pi_3(S^2) = \mathbb{Z}$$