Lecture 22 (3/8)

Tuesday, March 7, 2023 11:08 PM

$\boxed{\text{OH Thurs canceled}}$

$\pi_n(X) = \left\{ \begin{array}{c} \text{maps } f: S^n \to X \\ \text{up to homotopy} \end{array} \right\}$

Last time: • $\pi_n(X)$ is a group

• Commutative for $n > 1$.

Thm $\pi_n(S^n) = \mathbb{Z}$ for all n.

Idea: • $f: S^n \to S^n$, can define degree $(f) \in \mathbb{Z}$

• degree $(f \circ g) = \deg(f) + \deg(g)$, so this is a group homomorphism.

• If $f \sim g$ then degree $(f) =$ degree (g)

! so degree: $\pi_n(S^n) \longrightarrow \mathbb{Z}$ is well defined

• degree $(Id_{S^n}) = 1 \Rightarrow$ surjective

• Need to check it is injective, so arbitrary f is homotopic to a multiple of Id_{S^n}.

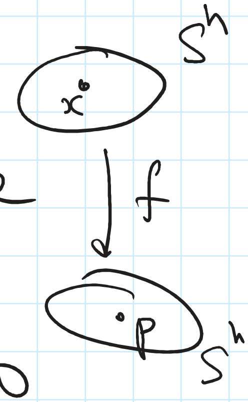
Definition of degree:

$f: S^n \longrightarrow S^n$. smooth

Sard's theorem: there is an open dense subset of regular values in S^n

regular values in S^n

<u>Recall:</u> $p \in S^n$ is a regular value if
$df : T_x S^n \longrightarrow T_p S^n$ is surjective
for all points $x \in f^{-1}(p)$.



df is <u>surjective</u> iff $\det(df) \neq 0$

In local coordinates $z_1 \ldots z_n$ in source
$$f(z_1 \ldots z_n) = (f_1(z_1 \ldots z_n), \ldots f_n(z_1 \ldots z_n))$$
df surjective iff $\det\left(\frac{\partial f_i}{\partial z_j}\right) \neq 0$.

Map of manifolds of same dimension
$\Rightarrow$ square matrix

<u>Fact</u> If p is a regular value then
$f^{-1}(p)$ = smooth manifold of dim $= n - n = 0$.
in this case

$\Rightarrow f^{-1}(p)$ = finite collection of points $x_1 \ldots x_k$

$$\boxed{\deg f = \sum_{i=1}^{k} \operatorname{sgn} \det(df_{x_i})}$$

$\neq 0$ since p is a regular value.

Alternatively, $\det(df_{x_i}) > 0$ if local orientation at x_i agrees under f with orientation at p

$\det(df_{x_i}) < 0$ if orientations do not agree

$\det(df_{x_i}) < 0$ if orientations do not agree.

df(positively oriented basis in $T_{x_i}(S^n)$) is a positively/negatively oriented basis in $T_p S^n$

Fact This does not depend on the choice of p.

Thm If f and g are homotopic then $\deg(f) = \deg(g)$.

Rmk All this make sense and Thm holds for maps $f: M^n \longrightarrow N^n$ of smooth oriented compact manifolds of same dimension.

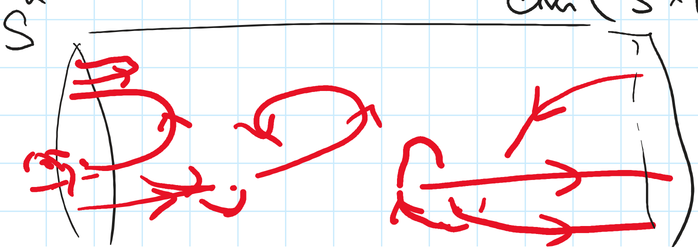
Proof of Thm: $F: S^n \times [0,1] \longrightarrow S^n$ homotopy

$F(x,0) = f(x) \qquad F(x,1) = g(x)$

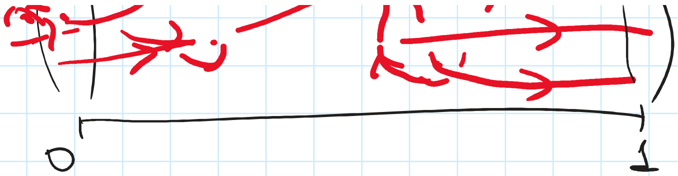
Can assume F is smooth, so we can again apply Sard's Theorem and find a regular value $p \in S^n$ for the map F.

$\Rightarrow F^{-1}(p)$ is a smooth submanifold of $S^n \times [0,1]$ of dimension $\underbrace{(n+1)}_{\dim(S^n \times [0,1])} - n = 1$, possibly with boundary.

S^n



$F^{-1}(p)$ is a union of circles and intervals.



circles and intervals.

oriented!

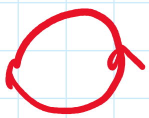
$G \cap (S^n \times \{0\}) = f^{-1}(p) =$ finite number of points in S^n

$G \cap (S^n \times \{1\}) = g^{-1}(p)$

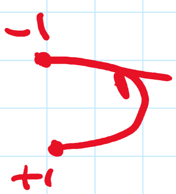
We claim that in this picture $\sum \text{sgn}(\det df) = \sum \text{sgn}(\det dg)$

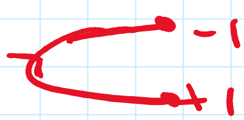
$\#$points on left with signs $= \deg f$

$\deg g = \#$points on the right with signs.

-  circle contributes 0 to both bdries

-  Interval connecting left bdry to right boundary $\to$ 1 point each

-  start and end on the left $\Rightarrow (+1) + (-1) = 0$ cancel.

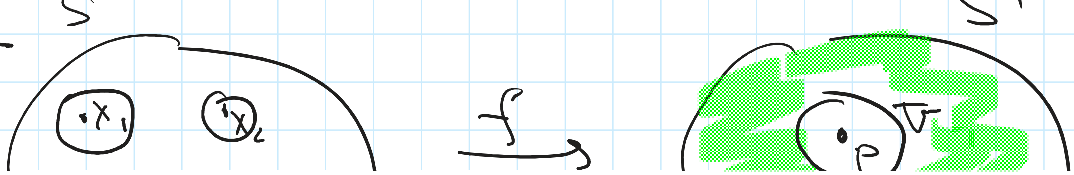
-  $\Rightarrow (+1) + (-1) = 0$ cancel.

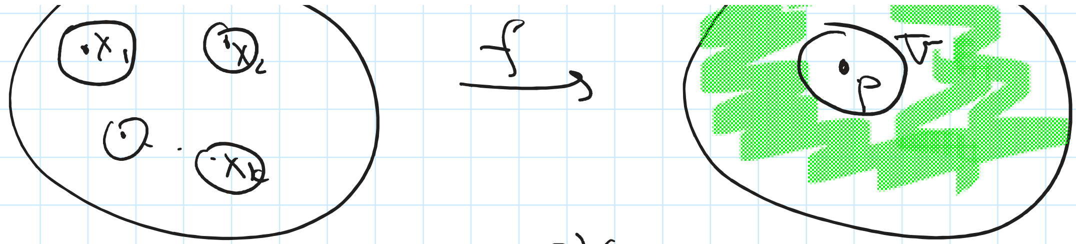
$\blacksquare$

$\deg : \pi_n(S^n) \longrightarrow \mathbb{Z}$ is well defined homomorphism.

Thm $f : S^n \to S^n$ arbitrary, then f is homotopic to a multiple of $[\mathrm{Id}_{S^n}]$ in $\pi_n(S^n)$

Proof:

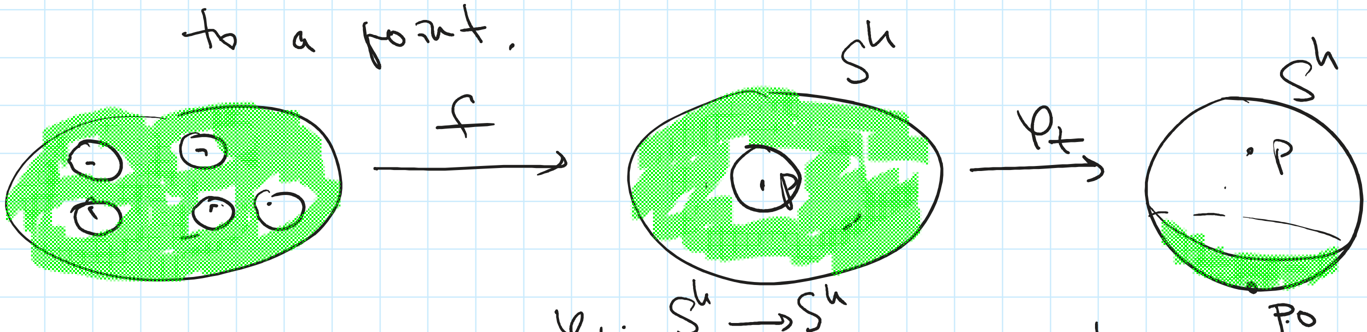




p = regular value, $f^{-1}(U)$ = collection of nbhds of x_i

U = nbhd of p (locally f is a diffeomorphism) by Inverse Function Thm.

Idea: we can shrink the complement $S^n \setminus U \cong D^n$ to a point.



$\varphi_t : S^n \to S^n$

homotopy shrinking $S^n - U$ to $p_0 \neq p$

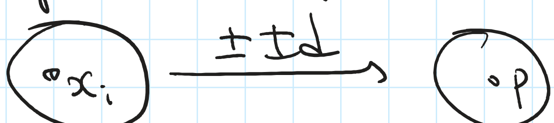
$\varphi_0 = Id_{S^n}$, φ_1 maps $S^n - U$ to p_0

$\varphi_t \circ f$ is a homotopy between $\varphi_0 \circ f = f$ and $\varphi_1 \circ f$.

At $t=1$ we map neighborhoods of x_i to $(S^n \setminus p_0) \sim U$

complement of all these to p_0

So $\varphi_1 \circ f$ = composition in $\pi_n(S^n)$ of standard maps



degree ± 1 map $\underset{S^n \cong}{D^n/\partial D^n} \longrightarrow \underset{S^n \cong}{D^n/\partial D^n}$

Conclusion: f is homotopic to $[\pm 1] + [\pm 1] + [\pm 1] + \dots$ in $\pi_n(S^n)$

Conclusion: f is homotopic to $\underbrace{[\pm 1] + [\pm 1] + [\pm 1] + \ldots}_{= [\deg f]}$ in $\pi_n(S^n)$

$\blacksquare$

<u>Rmk</u> Same argument works for maps $M^n \longrightarrow S^n$
These are classified by degree.