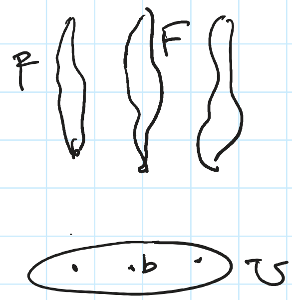
Lecture 23 (3/10)

Friday, March 10, 2023 9:56 AM

Locally trivial fibrations

Def: $p: E \to B$ is called a locally trivial fibration with fiber F, if for every point $b \in B$ there is a neighborhood $U \ni b$ such that $p^{-1}(U) \simeq U \times F$, more precisely, we have a commutative diagram



$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\simeq"] \arrow[dr, "p"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

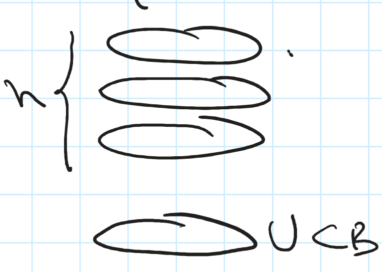
$(x, f) \mapsto x$

In particular, $p^{-1}(b) = F$ for all $b \in B$

Examples: ① Any covering is a locally trivial fibration

$F = \{\text{discrete set of points}\}$.

B = base
E = total space
F = fiber



$p^{-1}(U) = $ union of V_i, each $V_i \simeq U$

$= U \times \{n \text{ points}\}$

② M = smooth manifold, TM = tangent bundle of M

$= (p, v)$: p = point in M, v = tangent vector in T_pM

$\pi: TM \longrightarrow M$

$(p; v) \longrightarrow p$

This is a locally trivial fibration with fiber $\mathbb{R}^n$ ($n = \dim M$).

<u>Proof:</u> choose a chart on M with local coords $U \simeq$ open in $\mathbb{R}^n$

$$\pi^{-1}(U) = T\left(\text{open subset in } \mathbb{R}^n\right) \simeq \left(\text{open subset in } \mathbb{R}^n\right) \times \mathbb{R}^n.$$

<u>Warning/more next week</u> In general, $TM \not\simeq M \times \mathbb{R}^n$ so the tangent bundle is not <u>globally</u> trivial.

<u>Idea:</u> If $TM \simeq M \times \mathbb{R}^n$, we can pick a nonzero vector $v \in \mathbb{R}^n \rightsquigarrow$ vector field on M which is nonzero at every point.

<u>Fact:</u> Any vector field on S^2 has a <u>zero</u>.

$$\Rightarrow TS^2 \not\simeq S^2 \times \mathbb{R}^2.$$

<u>Rmk</u> $TM \simeq M \times \mathbb{R}^n$ iff we can find n vector fields on M which are linearly independent at every point (M parallelizable).

! <u>Rmk</u> From topological point of view, TM is homotopy equivalent to M $\quad (p,v) \overset{t=1}{\sim} (p,tv) \xrightarrow{t \to 0} (p,0)$

You can build some interesting spaces:

$\{(p,v) : p$ point in M, $v =$ <u>nonzero</u> tangent vector at $p\}$

$\parallel$

locally trivial fibration with fiber $\mathbb{R}^4 \setminus \{0\}$

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③ <u>Hopf fibration</u> $p: S^3 \longrightarrow \mathbb{CP}^1 = S^2$ $\{$complex lines in $\mathbb{C}^2\}$

$$S^3 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2 = \mathbb{R}^4$$

$$(z_1, z_2) \longrightarrow [z_1 : z_2]$$

<u>Claim</u> p is a locally trivial fibration with fiber S^1.

<u>Proof:</u> $U = \{z_1 \neq 0\}$, let's prove $p^{-1}(U) \simeq U \times S^1$.

Recall on $U = \{z_1 \neq 0\}$ we can write $[z_1 : z_2] \sim [1 : \frac{z_2}{z_1}]$

$U \simeq \mathbb{C}$ with coordinate w. $\qquad [1 : \frac{z_2}{z_1}] = [1 : w]$

Given $[1 : w]$, the preimage

$$p^{-1}([1:w]) = \{(z_1, z_2) : (z_1, z_2) = (\lambda, \lambda w) \text{ for some } \lambda \neq 0 \text{ and } |z_1|^2 + |z_2^2| = 1\}$$

$$|\lambda|^2 + |\lambda w|^2 = |\lambda|^2 + |\lambda|^2 |w|^2 = |\lambda|^2 \cdot (1 + |w|^2) = 1$$

$$\Longleftrightarrow |\lambda| = \frac{1}{\sqrt{1 + |w|^2}}$$

$$\underset{[1:w]}{U} \times \underset{\substack{\text{unit circle} \\ |\theta| = 1}}{S^1} \longrightarrow p^{-1}(U)$$

$$([1:w], \theta) \longrightarrow (\lambda, \lambda w) = \left(\frac{\theta}{\sqrt{1+|w|^2}}, \frac{\theta w}{\sqrt{1+|w|^2}} \right)$$

where $\lambda = \frac{\theta}{\sqrt{1+|w|^2}}$

Inverse map.

$$w = \frac{z_2}{z_1} \longleftarrow (z_1, z_2)$$

$\theta = z_1\sqrt{1+|w|^2} = z_1\sqrt{1+\left|\frac{z_2}{z_1}\right|^2}$

Similarly, $f^{-1}(z_2 \neq 0) = \underset{\mathbb{CP}^1}{\{z_2 \neq 0\}} \times S^1$,

so we cover $\mathbb{CP}^1$ by two charts, p is trivial on the preimage of both charts $\Rightarrow$ locally trivial fibration.

$\blacksquare$

<u>Claim</u> $S^3 \not\cong S^2 \times S^1$, so this is <u>not</u> globally trivial.

$\pi_1(S^3) = \{e\}$ $\quad \pi_1(S^2 \times S^1) = \pi_1(S^2) \times \pi_1(S^1) = \{e\} \times \mathbb{Z} = \mathbb{Z}$.

<u>Generalizations</u> of Hopf fibration:

- $(\mathbb{C}^2 \setminus \{0\}) \longrightarrow \mathbb{CP}^1$ with fiber $\mathbb{C}^*$.
- $S^{2n-1} = \left(\begin{array}{c}\text{unit sphere}\\ \text{in } \mathbb{C}^n\end{array}\right) \longrightarrow \mathbb{CP}^{n-1}$

  $(z_1 \ldots z_n) \longrightarrow [z_1 : \ldots : z_n]$

  $|z_1|^2 + \ldots |z_n|^2 = 1$

  locally trivial fibration with fiber S^1.

  $\mathbb{C}^n \setminus \{0\} \xrightarrow[\text{with fiber } \mathbb{C}^*]{} \mathbb{CP}^{n-1}$.
- $S^{4n-1} \longrightarrow \mathbb{HP}^{n-1}$ — quaternionic projective space

  with fiber S^3

  $S^7 \longrightarrow S^4 = \mathbb{HP}^1$ with fiber S^3.

⑤ $SO(n) \xrightarrow{\pi} S^{n-1}$ $\quad$ $n \geq 2$

$SO(n)$ = $n \times n$ orthogonal matrices, det. $= 1$ $\longleftrightarrow$ columns form an orthonormal basis in $\mathbb{R}^n$

$$A \longrightarrow (\text{first column of } A) = Ae_1$$

__Claim:__ This is a locally trivial fibration with fiber $SO(n-1)$.

__Proof__ First, let's check $\pi^{-1}(e_1) = SO(n-1)$.

$$\pi^{-1}(e_1) = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & \vdots & & \vdots \\ \vdots & & & \\ 0 & * & \cdots & * \end{pmatrix} = A \text{ orthogonal}$$

$A = (A_1 \ldots A_n)$

$A_i \perp A_1 \quad i \neq 1$

$\Rightarrow A_i = (0, B_i)$

B_i = length $(n-1)$ vector.

columns form orthonormal basis

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boxed{B} & \\ 0 & & & \end{pmatrix}$$

B orthogonal $(n-1) \times (n-1)$ matrix

$\det B = \det A = 1$.

Equivalently, we have a group action $SO(n)$ on $\mathbb{R}^n$

Orbit of $e_1 = S^{n-1}$

Stabilizer of $e_1 = \{A : Ae_1 = e_1\} = SO(n-1)$.