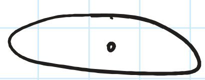
Locally trivial fibrations

$E \xrightarrow{p} B$ For every point $b \in B$ there is a neighborhood U such that $p^{-1}(U) = U \times F$

total space, base

F  F = fiber $:= p^{-1}(b)$

Globally, $E \neq B \times F$

 U

Ex: Hopf fibration $S^3 \longrightarrow S^2$, fiber $= S^1$

Ex: $SO(n) \xrightarrow{\pi} S^{n-1}$

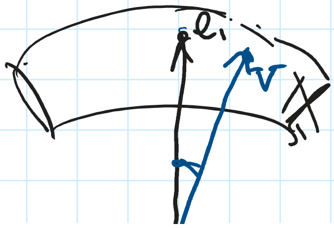
$n \times n$ orthogonal matrices $\det = 1$

$A \longrightarrow Ae_1 =$ first column of A.

Claim This is a locally trivial fibration with fiber $SO(n-1)$.

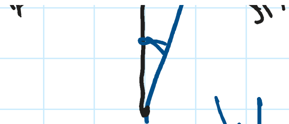
Proof: 1) Locally trivial near e_1, that is, there is a neighborhood $U \ni e_1$ such that

$\pi^{-1}(U) = \underline{U \times SO(n-1)}$.



v = vector, $|v| = 1 \Rightarrow v \in S^{n-1}$

v is close to e_1



v is close to e_1

We can choose a specific orthogonal matrix A_v which sends e_1 to v

- Choose a 2-dim plane P_v containing e_1 and v
- Choose orthonormal basis in $P_v^\perp$
- A_v = rotation by smallest angle in P_v which sends e_1 to v, and identity on $P_v^\perp$.

This A_v depends continuously on v as long as the angle between e_1 and v is $< \pi$.

Define a map $U \times SO(n-1) \longrightarrow SO(n)$

$$\underbrace{(v}_{\text{vector}}, B) \xrightarrow{\varphi} A_v \cdot \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{B} \\ 0 & \end{pmatrix}$$

$$A_v \cdot \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{B} \\ 0 & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = A_v \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = v$$

Conversely, suppose $Ae_1 = v$ for some $A \in SO(n)$

Then $(A_v^{-1} \cdot A) \cdot e_1 = A_v^{-1}(v) = e_1$

$$\Rightarrow A_v^{-1} \cdot A = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{B} \\ 0 & \end{pmatrix} \text{ for some } B \in SO(n-1)$$

and

$$A = A_v \cdot \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & B \end{pmatrix}$$

$$A = A_v \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{B} \\ 0 & \end{pmatrix}.$$

In other words, we get a bijection/homeomorphism

$$\left\{ \begin{array}{c} \text{all matrices} \\ A \text{ such that} \\ Ae_1 = v \end{array} \right\} \xleftrightarrow{A_v} \left\{ \begin{array}{c} \text{stabilizer} \\ \text{of } e_1 \end{array} \right\} = SO(n-1)$$

This proves $\pi^{-1}(U) \simeq U \times SO(n-1)$

$$A_v \cdot B \longleftarrow (v, B)$$

$$A \longrightarrow (\underbrace{Ae_1}_{v}, \ A_v^{-1} \cdot A)$$

(2) Now we can do this for any other point on S^{n-1}

$x \in S^{n-1}$, choose some $A_x \in SO(n)$

such that $A_x(e_1) = x$

Consider new neighborhood $A_x(U) = U_x$

which consists of vectors $A_x(v) = A_x \cdot A_v(e_1)$

for $v \in U$.

& repeat same argument. ∎

HW Same idea for $SU(n) \longrightarrow S^{2n-1}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\qquad\qquad\qquad\qquad$ unit sphere in $\mathbb{C}^n = \mathbb{R}^{2n}$

$$A \longrightarrow Ae_1$$

You need to find an analogue of A_v for $v \in S^{2n-1}$

The rest of argument goes through. close to e_1

Lots of similar fibrations from group actions

Ex $Gr(k,n) = \{$all k-dimensional planes in $\mathbb{R}^n$ through $0\}$ = Grassmannian

$Gr(1,n) = \{$lines through 0 in $\mathbb{R}^n\} = \mathbb{R}P^{n-1}$

$SO(n)$ acts on the set of all k-planes.

$SO(n) \longrightarrow Gr(k,n)$

$A \longrightarrow$ Span(first k columns of A) $= A\langle e_1 \dots e_k\rangle$

Claim This is a locally trivial fibration, fiber = stabilizer of Span$\langle e_1 \dots e_k\rangle$

$$\left\{ A = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right) \right\} = SO(k)\times SO(n-k).$$

How can we use locally trivial fibrations to study π_1 or π_k ?

Def A long exact sequence of groups is

Def A long exact sequence of groups is a sequence $\to G_i \xrightarrow{\varphi_i} G_{i-1} \xrightarrow{\varphi_{i-1}} G_{i-2} \to \ldots$

G_i = some groups (not necessarily abelian)

φ_i = group homomorphisms

$$\boxed{\operatorname{Ker}\varphi_{i-1} = \operatorname{Im}\varphi_i \quad \text{for all } i}$$

$(\Rightarrow \varphi_{i-1} \circ \varphi_i = e)$

Thm (Long exact sequence of fibration)

Suppose $E \xrightarrow{p} B$ is a locally trivial fibration with fiber F, and • E and B path connected
• E, B, F = CW complexes.

Then we have a long exact sequence:

$$\ldots \longrightarrow \pi_k(F) \longrightarrow \pi_k(E) \xrightarrow{p_*} \pi_k(B) \longrightarrow$$
$$\to \pi_{k-1}(F) \to \pi_{k-1}(E) \to \pi_{k-1}(B)$$
$$\to \ldots \to \pi_1(F) \to \pi_1(E) \to \pi_1(B)$$

Example: Hopf fibration $E = S^3, B = S^2, F = S^1$

$$\ldots \to \pi_4(S^1) \to \pi_4(S^3) \to \pi_4(S^2) \to$$
$$\to \pi_3(S^1) \to \pi_3(S^3) \to \pi_3(S^2) \to$$

$\pi_3(S^3) = \mathbb{Z}$

$\cdots \pi_4(S^1) \to \pi_4(S^3) \to \pi_4(S^2) \to \cdots$ $\qquad \pi_3(S^3) = \mathbb{Z}$

$$\begin{array}{l}
\to \pi_3(S^1) \to \pi_3(S^3) \to \pi_3(S^2) \to \\
\to \pi_2(S^1) \to \pi_2(S^3) \to \pi_2(S^2) \to \\
\to \pi_1(S^1) \to \pi_1(S^3) \to \pi_1(S^2)
\end{array}$$

$\pi_2(S^2) = \mathbb{Z}$

$\pi_1(S^1) = \mathbb{Z}$

$\pi_1(S^k) = 0$ for $k > 1$

$\pi_2(S^3) = 0$

$\pi_k(S^1) = 0$ for $\underline{k > 1}$ $\qquad \pi_k(S^n) = 0$ if $\underline{k < n}$

$\pi_k(X) = \pi_k(\tilde{X})$ if $\tilde{X} \to X$ covering map

$\pi_k(S^1) = \pi_k(\mathbb{R}) = 0$

$k > 1$

$$\begin{array}{l}
\to 0 \to \pi_4(S^3) \to \pi_4(S^2) \to \\
\to 0 \to \mathbb{Z} = \pi_3(S^3) \to \pi_3(S^2) \to \\
\to 0 \to 0 \to \underbrace{\mathbb{Z}}_{\pi_2(S^2)} \to \\
\to \underbrace{\mathbb{Z}}_{\pi_1(S^1)} \to 0 \to 0
\end{array}$$

<u>Conclusion</u>: 1) $0 \to \mathbb{Z} \to \pi_3(S^2) \to 0$

So $\boxed{\pi_3(S^2) \simeq \mathbb{Z}.}$ (and the generator is the generator is the Hopf fibration map $S^3 \to S^2$)

2) $0 \longrightarrow \pi_4(S^3) \longrightarrow \pi_4(S^2) \longrightarrow 0$

so $\pi_4(S^3) \simeq \pi_4(S^2)$ and, similarly,

$\pi_k(S^3) \simeq \pi_k(S^2)$ for all $k \geq 3$.