Lecture 25 (3/16)

Monday, March 13, 2023 1:05 PM

Homotopy groups of spheres

$\pi_1(S^1) = \mathbb{Z}$

$\pi_3(S^2) = \mathbb{Z}$ Hopf fibration

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z2 2	$Z_{12}×Z_2$	Z_{84}×Z2 2	Z2 2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z2 2	$Z_{12}×Z_2$	Z_{84}×Z2 2	Z2 2
S^4	0	0	0	Z	Z_2	Z_2	$Z×Z_{12}$	Z2 2	Z2 2	$Z_{24}×Z_3$	Z_{15}	Z_2	Z3 2	Z_{120}× $Z_{12}×Z_2$	Z_{84}×Z5 2
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z3 2	$Z_{72}×Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}×Z_2$	Z3 2
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z3 2
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z×Z_{120}$

From <https://en.wikipedia.org/wiki/Homotopy_groups_of_spheres>

$\pi_1(S^k) = 0$ if $k > 1$

all 0 below diagonal

$\pi_k(S^n) = 0$ if $n > k$

$\pi_k(S^k) = \mathbb{Z}$ diagonal

$\pi_k(S^1) = 0$, $k > 1$

Fact

- $\pi_3(S^2) = \mathbb{Z}$ (Hopf fibration)
- $\pi_{k+1}(S^k) = \mathbb{Z}_2$ for $k \geq 3$
  $\qquad \shortparallel$
  $\pi_4(S^3), \pi_5(S^4), \dots$
- For fixed i and k large enough, $\pi_{k+i}(S^k)$ stabilize to "stable homotopy groups"

So for $k \gg 0$ we get $\pi_k(S^k) = \mathbb{Z}$
$\pi_{k+1}(S^k) = \mathbb{Z}_2$

$\pi_{k+2}(S^k) = \mathbb{Z}_2$

$\pi_{k+3}(S^k) = \mathbb{Z}_{24}$

$\vdots$

Thm (Serre) $\pi_n(S^k)$ has a nontrivial free part

if $(n,k) = (n,n)$ or $(n,k) = (4i-1, 2i)$
$n = k$

$\pi_{4i-1}(S^{2i})$ has free part

$i=1 \quad \pi_3(S^2)$
$i=2 \quad \pi_7(S^4)$
$i=3 \quad \pi_{11}(S^6)$
$\dots$

(hard)

Goal for today: $\pi_1(SO(n)) = ?$

(HW) $\pi_1(SU(n))$ very similar parallel.

(1) What happens for small n?

$SO(1) = \{(1)\} = pt \qquad \pi_1 = 0$

$SO(2) = \{\text{rotations of } \mathbb{R}^2\} = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \; 0 \le \varphi \le 2\pi \right\}$

Topologically, $\varphi \in S^1$, so $\boxed{SO(2) \simeq S^1}$

$\pi_1(SO(2)) = \mathbb{Z}$.

$SU(1) = \{(1)\} \quad \pi_1 = 0 \qquad U(1) = (e^{i\varphi}) = S^1$

$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \quad \begin{matrix} \alpha, \beta \in \mathbb{C} \\ |\alpha|^2 + |\beta|^2 = 1 \end{matrix} \right\}$

$$\left\{ \begin{pmatrix} \ & \ \\ \beta & \bar\alpha \end{pmatrix} \quad |\alpha|^2+|\beta|^2=1 \right\}$$

Let's check this is in $SU(2)$: $\alpha\bar\alpha+\beta\bar\beta=1$

- Colums form a Hermitian orthonormal basis

$$\left\langle \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right\rangle = \bar\alpha\alpha+\bar\beta\beta=1$$

$$\left\langle \begin{pmatrix}-\bar\beta\\ \bar\alpha\end{pmatrix}, \begin{pmatrix}-\bar\beta\\ \bar\alpha\end{pmatrix}\right\rangle = (-\beta)\cdot(-\bar\beta)+(\alpha)\cdot(\bar\alpha)=$$

$$=\beta\bar\beta+\alpha\bar\alpha=1.$$

$$\left\langle \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}-\bar\beta\\ \bar\alpha\end{pmatrix}\right\rangle = \bar\alpha(-\bar\beta)+\bar\beta\cdot\bar\alpha=0.$$

Can also compute $\bar{A}^{t}A=I$

- $\det A=\alpha\bar\alpha+\beta\bar\beta=1.$

Any 2×2 matrix in $SU(2)$ has this form:

Hint: first prove $A=\begin{pmatrix}\alpha & -k\bar\beta\\ \beta & k\bar\alpha\end{pmatrix}$ for some k (from orthogonality of columns)

Use det to prove $k=1$....

Topologically, $SU(2)\simeq S^3$!

In particular, $\pi_1(SU(2))=\pi_1(S^3)=0$.

Fact: There is a surjective homomorphism

$SU(2)\longrightarrow SO(3)$ with kernel $\{\pm I\}$, so

$$\boxed{SO(3)=SU(2)/\pm I.}$$

Topologically $SU(2)/\ \ \ S^3/\ = \mathbb{RP}^3$

Topologically, $SU(2)/\pm I = S^3/\pm I = \mathbb{R}P^3$

Therefore, $\boxed{SO(3) \simeq \mathbb{R}P^3}$

In particular, $\boxed{\pi_1(SO(3)) = \mathbb{Z}_2}$

Rmk We can choose an axis L in S^3,

and consider a loop of rotations by angles $0 \le \varphi \le 2\pi$

around L.

(For $L = z$-axis we get $\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$)

This loop is nontrivial in $\pi_1(SO(3))$ and generates π_1.

Note Choice of axis L does not matter because the space of all axis is $\mathbb{R}P^2 \leftrightarrow$ connected.

Thm $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$.

Proof: Recall locally trivial fibration

$SO(n) \longrightarrow S^{n-1}$

$SO(n-1)$

$E = SO(n) \quad B = S^{n-1} \quad F = SO(n-1)$

Long exact sequence of fibration:

$$\cdots \to \pi_2(F) \to \pi_2(E) \to \pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B)$$

$$\cdots \pi_2(SO(n-1)) \to \pi_2(SO(n)) \to \underset{\substack{\| \\ 0}}{\pi_2(S^{n-1})} \to \pi_1(SO(n-1)) \to \pi_1(SO(n)) \to \pi_1(S^{n-1}) = 0$$

Assume $\underline{n \geq 3}$, then $n-1 \geq 2$, so $\pi_2(S^{n-1}) = 0$

$\pi_1(S^{n-1}) = 0$

Therefore $\pi_1(SO(n-1)) \simeq \pi_1(SO(n))$

$$\pi_1(SO(n)) = \pi_1(SO(n-1)) = \pi_1(SO(n-2)) \cdots = \underset{\substack{\| \\ \mathbb{Z}_2}}{\pi_1(SO(3))}$$

So $\pi_1(SO(n)) \simeq \pi_1(SO(3)) = \mathbb{Z}_2$ for $n \geq 3$.

<u>Cor</u> $SO(n)$ has an interesting simply connected double cover (spin group).

HW Need to do the same for $SU(n)$:

locally trivial fibration $SU(n) \longrightarrow S^{2n-1}$

(similar to last lecture) $A \longrightarrow$ first column of A

fiber $= SU(n-1)$

$$\to \pi_2(SU(n-1)) \to \pi_2(SU(n)) \to \pi_2(S^{2n-1})$$

$$\to \pi_1(SU(n-1)) \to \pi_1(SU(n)) \to$$
$$\to \pi_1(S^{2n-1}).$$

$\cdots$

$SO(2) = S^1$

Ex $SO(3) \xrightarrow[SO(2)]{} S^2$

$$\to \pi_2(SO(2)) \to \pi_2(SO(3)) \to \pi_2(S^2) \to$$

$\pi_2(SO(2)) = 0$, $\pi_2(SO(3)) = 0$, $\pi_2(S^2) = \mathbb{Z}$

$$\to \pi_1(SO(2)) \to \pi_1(SO(3)) \to \pi_1(S^2)$$

$\pi_1(SO(2)) = \mathbb{Z}$, $\pi_1(S^2) = 0$

$SO(3) = \mathbb{RP}^3$

$\pi_2(SO(3)) = \pi_2(\mathbb{RP}^3) \underset{\text{covering}}{=} \pi_2(S^3) = 0$

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \pi_1(SO(3)) \to 0$$

$\pi_1(SO(3)) = \mathbb{Z}_2$

So $\pi_2(S^2) \to \pi_1(SO(2))$ is nonzero and given by multiplication by 2.

Rmk $\pi_1(SO(n)) = \mathbb{Z}_2$ is related to $\pi_{n+1}(S^n) = \mathbb{Z}_2$ for large n.