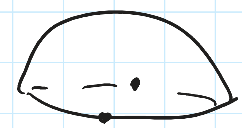
Lecture 26 (3/17)

Friday, March 17, 2023 10:05 AM

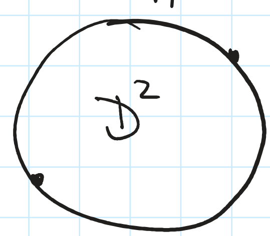
① $SO(3) \simeq \mathbb{R}P^3$

__Recall__ $\mathbb{R}P^n = S^n / \pm 1 = D^n / \pm 1 \text{ on } \partial D^n = S^{n-1}$

$\mathbb{R}P^2 =$  $S^2 / \pm 1 = D^2 / \pm \text{ on } \partial D^2$ 

upper half of S^2

$\|$



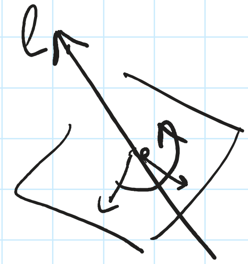
$\mathbb{R}P^3 = D^3 / \pm 1 \text{ on } \partial D^3 \overset{?}{=} SO(3)$

__Fact__ Any element of $SO(3)$ is a rotation around some axis in $\mathbb{R}^3$

(axis = eigenvector with eigenvalue 1)

$A \in SO(3) \longleftrightarrow$ rotation around some axis ℓ, by angle φ

Counter-clockw.



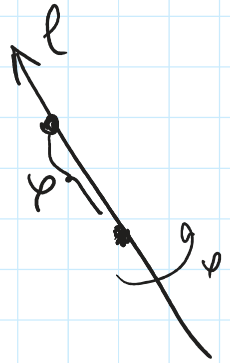
we want to __orient ℓ__ and say that we rotate by $\underline{0 \le \varphi \le \pi}$ counter-clockwise if we look from positive direction.

__Note:__ If angle $> \pi$, reverse orientation on ℓ.

Issue : $(\ell, \pi) \sim (-\ell$ ...th ... $-\pi)$

opposite orienten

<u>Construction</u> $A \in SO(3) \rightsquigarrow (\ell, \varphi) \rightsquigarrow$



$\rightsquigarrow$ point at a distance φ on the line ℓ. $(*)$

$0 \le \varphi \le \pi$

Almost a bijection between

$SO(3) \longleftrightarrow$ all such points.

<u>Special cases</u>: $\varphi = 0 \implies A = I \implies$ point is 0 <u>OK</u>.
does not depend on ℓ

$\varphi = \pi \implies (\ell, \pi) \sim (-\ell, -\pi)$
ℓ with opposite orientation

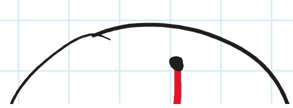
$(*)$: $SO(3) \longrightarrow B^3_\pi$ 3-ball with radius π

$\varphi = 0 \longrightarrow$ center of the ball

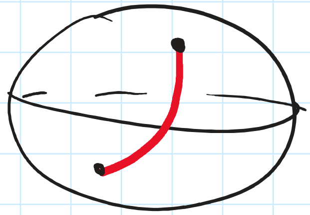
$\varphi = \pi$, $\ell \sim -\ell \longrightarrow B^3_\pi \big/ {\pm 1 \text{ on } \partial B^3} = \mathbb{RP}^3$.

<u>Application</u> $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$, how to find a
$\pi_1(SO(3))$ nontrivial loop in $SO(3)$?

$\mathbb{RP}^n = S^n / \pm 1$



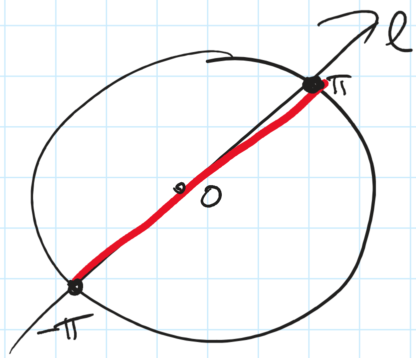
$= D^n / \pm \text{ on } \partial D^n$

$\mathbb{R}P^n = S^n/_{\pm 1}$  $= D^n/_{\pm \text{ on } \partial D^n}$ 

Connect opposite points by a path.

connect opposite pts on ∂D^n

$SO(3) = D^3_\pi / {\pm 1 \text{ on } \partial D^n}$



(l,π) and $(l,-\pi)$ are opposite pts on ∂D^3_π which project to same pt on $\mathbb{R}P^3$

$\underline{\{(l,\varphi) \mid -\pi \le \varphi \le \pi\}}$ is a loop in $\mathbb{R}P^3$ generating $\pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$.

$S^1 \to SO(3)$
φ

<u>Ex</u> Choose $l = z$-axis

$(l,\varphi) =$ rotation around z-axis by φ $= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$S^1 \longrightarrow SO(3)$

$\underline{\varphi = 0 \quad I}$ $\qquad \varphi = \pi$ or $-\pi$ $\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$

<u>Rmk</u> For $n \ge 3$, $\begin{pmatrix} \cos\varphi & -\sin\varphi & \\ \sin\varphi & \cos\varphi & 0 \\ & & \end{pmatrix} \in SO(n)$ $\quad S^1$

$$\begin{pmatrix} \sin\varphi & \cos\varphi & & \\ & & 1 & 0 \\ & 0 & & \ddots \\ & & & & 1 \end{pmatrix} \in SO(n)$$

This is a nontrivial loop in $SO(n)$ representing the generator in $\underbrace{\pi_1(SO(n)) = \mathbb{Z}_2}$.

proof: LES from last lecture.

Thm $T(S^2) \not\cong S^2 \times \mathbb{R}^2$

Sketch of proof Consider unit tangent bundle to S^2

$UT(S^2) = (p, v)$: v tangent vector at p to S^2, $|v| = 1$ (think of v as a vector in $\mathbb{R}^3$ perp. to radius)

Claim: $X = T(S^2) \setminus \{\text{zero section}\} =$

$$= \{(p, v) : v \text{ tangent vector at } p,\ v \neq 0\}$$

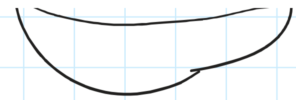
Then $X \sim UT(S^2)$ homotopy eq.

$$(p, v) \longrightarrow \left(p,\ tv + (1-t)\frac{v}{\|v\|}\right)$$

What is $UT(S^2)$?

Claim: $UT(S^2) = SO(3)$

Think of $p \in S^2$ as a vector in $\mathbb{R}^3$

Think of $p \in S^2$ as a vector in $\mathbb{R}^3$ 

$p \perp v$, $|p| = 1$, $|v| = 1$

Can find a unique w such that $w \perp p$, $w \perp v$

$|w| = 1$ and (w, p, v) is positively oriented

$$\rightsquigarrow A = \begin{pmatrix} | & | & | \\ w & p & v \\ | & | & | \end{pmatrix} \in SO(3)$$

<u>Proof of thm:</u> If $T(S^2) \simeq S^2 \times \mathbb{R}^2$ trivial bundle

$$\begin{tikzcd}
\text{then } X = \left\{ \begin{matrix} T(S^2) \\ v \neq 0 \end{matrix} \right\} \arrow[r, phantom, "\simeq"] & S^2 \times \{\mathbb{R}^2 - 0\} \\
UT(S^2) = SO(3) \arrow[u, hook] & S^2 \times S^1 \arrow[u, hook]
\end{tikzcd}$$

$$SO(3) \,\|\, \mathbb{R}P^3$$

Then $\mathbb{R}P^3 = SO(3)$ is homotopy equivalent to $S^2 \times S^1$

But $\pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$, $\pi_1(S^2 \times S^1) = \mathbb{Z}$,

so $\mathbb{R}P^3 \not\simeq S^2 \times S^1$, contradiction. $\blacksquare$

<u>Euler characteristic</u>

X = finite CW complex

$$\chi(X) = (\#\,0\text{-cells}) - (\#\,1\text{-cells}) + (\#\,2\text{-cells}) - \dots + (-1)^k (\#\,k\text{-cells}).$$

Facts: ① This does not depend on cell decomposition!

(proof: homology) and this is a homotopy invariant!

② $X = Z \cup W$, Z closed subcomplex in X

$W = X - Z$ open

$\chi(X) = \chi(Z) + \chi(W)$ clear (some cells in Z, some in W)

③ $\tilde{X} \longrightarrow X$ covering map of degree n

then $\chi(\tilde{X}) = n\,\chi(X)$

(for each cell in X, n cells in $\tilde{X}$ of same dimension).

④ $$\chi(S^n) = \underbrace{1}_{\text{0-cell}} + \underbrace{(-1)^n}_{\text{n-cell}} = \begin{cases} 2, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases}$$

$$\chi(RP^n) = \frac{1}{2}\chi(S^n) = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases}$$

$$\chi(\Sigma_g = \text{genus } g \text{ surface}) = 2 - 2g$$

(polygon picture, 1 0-cell

2g 1-cells

1 2-cell.

$1 - 2g + 1 = 2 - 2g.$

(5) M = smooth orientable manifold

v = vector field on M

$\Rightarrow$ $\boxed{\text{sum of indices of singular points of } M = \chi(M)}$

<u>Cor</u> If $\chi(M) \neq 0$, then <u>any</u> vector field on M must have a singular point.

($\Rightarrow TM \neq M \times \mathbb{R}^n$).