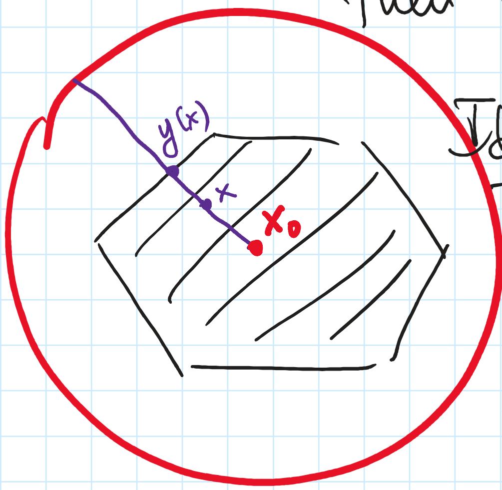
Lecture 3 (1/13)

Friday, January 13, 2023 9:54 AM

<u>Fact</u> P = compact convex polyhedron in $\mathbb{R}^n$ (w. interior)

then $P \approx D^n$, $\partial P \approx S^{n-1}$



Idea of proof: • choose a point x_0 in the interior of P

• Choose R large enough so that $D^n_R(x_0)$ = ball with center at x_0 radius R

$P \subset D^n_R(x_0)$

• Given a point $x \in P$, define $y(x) \in \partial P$ such that $y(x), x, x_0$ are collinear

• "stretch" the segment $[x_0, y(x)]$ so that it has length R:

$x \to f(x)$, $f(x)$ on same line as $x, x_0, y(x)$

$$\|f(x) - x_0\| = \frac{\|x - x_0\| \cdot R}{\|y(x) - x_0\|}$$

$\Rightarrow$ explicit fla for $f(x)$ (Do it!)

• Prove f is continuous, using that $y(x)$ is continuous !

• If $x \in \partial P$ then $x = y(x)$ and $f(x) \in S^{n-1}$

so this also gives a map $\partial P \longrightarrow S^{n-1}$

Restriction of a continuous map to a closed subset is continuous.

→ Hint: use this to prove that suspension of a cube is $\simeq D^n$ (HW #4)

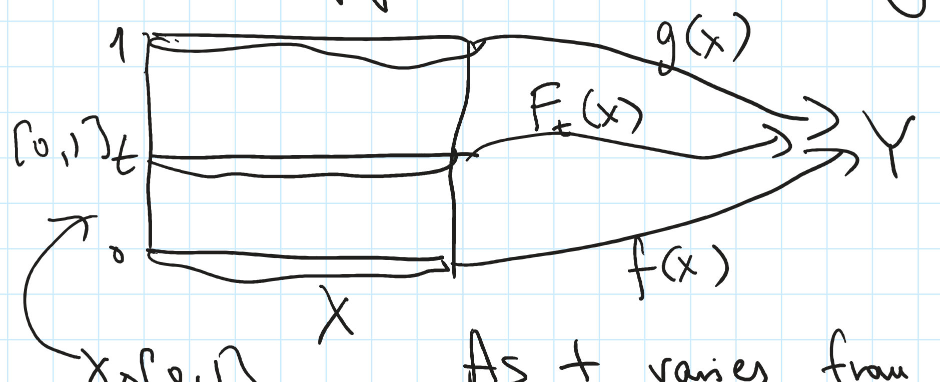
Def $f, g : X \longrightarrow Y$ two continuous maps

We say that f and g are homotopic if there is a continuous map

$$F : X \times [0,1] \longrightarrow Y \quad \text{such that}$$

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = g(x) \text{ for all } x \text{ in } X.$$

F = homotopy between f and g.



$F_t(x) = F(x,t)$ for some fixed $t \in [0,1]$.

As t varies from 0 to 1, we get a family of maps $F_t : X \longrightarrow Y$

$F_0 = f$, $F_1 = g$, continuous in t.

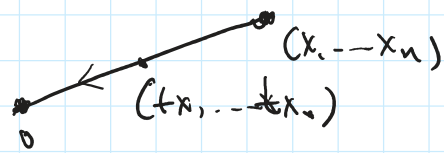
Ex $F : \mathbb{R}^n \times [0,1] \longrightarrow \mathbb{R}^n$

$$F(x_1, \ldots x_n; t) = (tx_1, tx_2, \ldots tx_n)$$

$$F_t(x_1 \ldots x_n) \longleftarrow \text{family of maps } \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$t=0$: $F_0(x_1 \ldots x_n) = (0, \ldots \ldots 0)$ $\quad \mathbb{R}^n \longrightarrow \{0\}$

$t=1$: $F_1(x_1 \ldots x_n) = (x_1 \ldots x_n)$ $\quad F_1 = Id_{\mathbb{R}^n}$



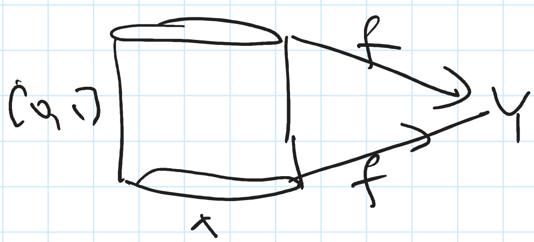
"straight line homotopy"

<u>Thm</u> (a) Homotopy is an equivalence relation on maps $X \to Y$

(b) $f, g : X \to Y$, $h : Y \to Z$ and $f \sim g$ (f is homotopic to g) $\Rightarrow$ $h \circ f \sim h \circ g$

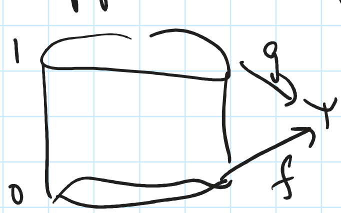
homotopic

<u>Proof</u> (a) • $f \sim f$



$F(x,t) = f(x)$ for all t

• Suppose $f \sim g$, and there is a homotopy



$F : X \times [0,1] \longrightarrow Y$

$F(x,0) = f(x) \quad F(x,1) = g(x)$

Consider $F' : X \times [0,1] \longrightarrow Y$

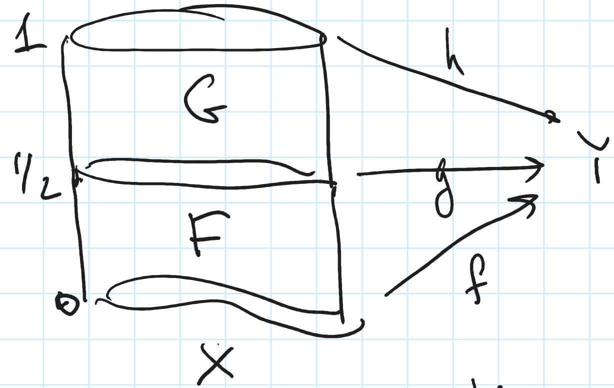
$$F'(x,t) = F(x, 1-t)$$

$F'(x,0) = F(x,1) = g(x) \qquad F'(x,1) = F(x,0) = f(x)$

So $g \sim f$.

- Suppose $f \sim g$, $g \sim h$, and there are homotopies $F : X \times [0,1] \longrightarrow Y$ $\quad F_0 = f \quad F_1 = g$

$\qquad G : X \times [0,1] \longrightarrow Y$, $\quad G_0 = g \quad G_1 = h$



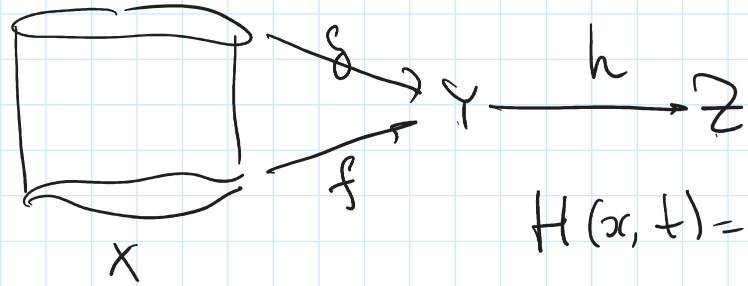
Define a new map $H : X \times [0,1] \longrightarrow Y$ such that:

$$H(x,t) = \begin{cases} F(x,2t), & 0 \le t \le \frac{1}{2} \\ G(x,2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

This is continuous, $H(x,0) = F(x,0) = f$

$H(x,1) = G(x,1) = h$ so $f \sim h$.

(b)



$f \sim g$, F = homotopy

$F_0 = f \quad F_1 = g$

$H(x,t) = h(F(x,t))$

continuous

$H(x,0) = h \circ f \qquad H(x,1) = h \circ g$.

Homotopy category: objects = top. spaces

morphisms = continuous maps

morphisms = $\underset{\sim}{\underline{\text{continuous maps}}}$

By thm, can pick any representative in eq. class of maps, composition is well defined

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Def A space X is contractible if the identity map Id_X is homotopic to a constant map $X \to \{p\}$ for some point $p \in X$.

Unpack: $F : X \times [0,1] \longrightarrow X$

$$F(x,0) = x \qquad F(x,1) = p \text{ for all } x$$

Informally, we "shrink" the space X into one point p.

Ex $F(x_1, \dots x_n; t) = (tx_1 \dots tx_n)$

$$Id_{\mathbb{R}^n} \simeq \{0\} \implies \mathbb{R}^n \text{ is contractible.}$$

Ex D^n is contractible, any convex subset $P \subset \mathbb{R}^n$ such that $\bar{0} \in P$ is contractible by the same formula.

Note: Since P is convex, $(x_1 \dots x_n) \in P \ni (0 \dots 0)$

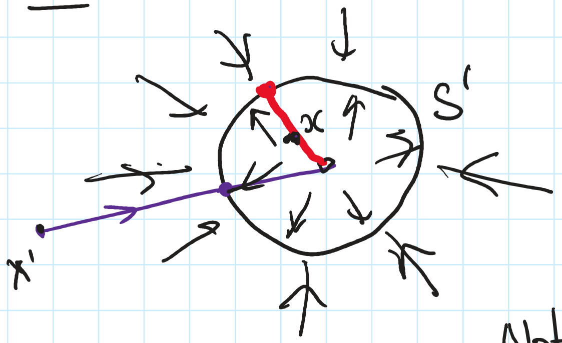
$$\implies (tx_1 \dots tx_n) \in P.$$

Fact: S^1 is NOT contractible!

Fact: S is NOT contractible!

S^n is NOT contractible!

Ex: $\mathbb{R}^2 \setminus \{0\} = X \supset S^1 = \{x : \|x\| = 1\}$



We can shrink X to S^1:

start with $F_0(x) = x$

finish with $F_1(x) = \dfrac{x}{\|x\|}$

Note: This is well defined since $x \neq 0$!

We can do this continuously:

$$F_t(x) = t \cdot \frac{x}{\|x\|} + (1-t)\,x$$

Continuous in x and in t, $t=0 \Rightarrow F_0(x) = x$

$t=1 \Rightarrow F_1(x) = \dfrac{x}{\|x\|}$.