Lecture 4 (1/18)

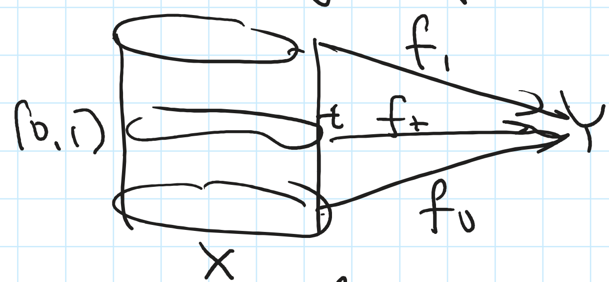
Monday, January 16, 2023 2:14 PM

Recap: homotopy

$F: X \times [0,1] \to Y$ continuous

$f_t : X \to Y \qquad f_t(x) = f(x,t) \qquad t \in [0,1]$

family of maps continuously interpolating between f_0 and f_1:



① Cellular Approximation Theorem

$X, Y =$ cell complexes

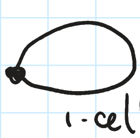
$f: X \to Y$ is called *cellular* if

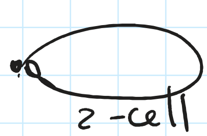
$f(X^k) \subset Y^k$ for all k

k-skeleton

Thm (Hatcher 4.8) Any continuous map $f: X \to Y$ is homotopic to a cellular map. (w/o proof)

Ex $X = S^1 \qquad Y = S^2$





f cellular $: S^1 \to S^2$ if $f(\text{0-cell}) = \text{0-cell}$

$f(\text{1-cell}) \subset \text{1-skeleton}(S^2)$

$= \text{0-cell}$

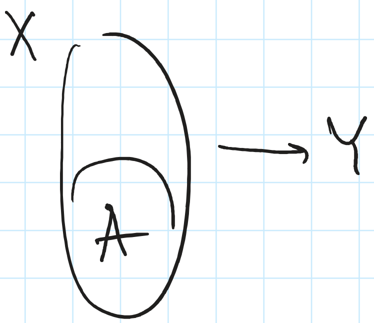
$\Rightarrow f =$ constant, sends $S^1 \to$ 0-cell

Cor Any continuous function $S^1 \to S^2$ is homotopic to a constant map.

Note There are really bad (ex. "surjective") "space-filling" continuous maps $S^1 \to S^2$.

② Homotopy extension

X = cell complex $\qquad A$ = subcomplex (= closed union of cells)



f_0 : given function $X \to Y$

$g_0 = f_0|_A : A \to Y$

Thm Suppose that we have a homotopy $g_t : A \to Y$. Then it extends to a homotopy $f_t : X \to Y$ such that f_0 agrees, and $f_t|_A = g_t$.

Proof: next time

Def X, Y = two top. spaces they are called homotopy equivalent if there are maps (continuous)

$f : X \to Y$, $g : Y \to X$ such that

$f \circ g \sim Id_Y \qquad g \circ f \sim Id_X$

homotopic

homotopic

Ex 1 Recall: X is contractible if $\mathrm{Id}_X \sim$ constant map

X is contractible iff X is homotopy equivalent to a point.

"can shrink X into one pt"

Proof: Suppose $\mathrm{Id}_X \sim \varphi$ $\quad \varphi : X \to \{x_0\}$

$\varphi(x) = x_0$ for all x

$$\begin{tikzcd} X \arrow[r, bend left, "\varphi"] & \{x_0\} \arrow[l, bend left, "\mathrm{Id}"] \end{tikzcd}$$

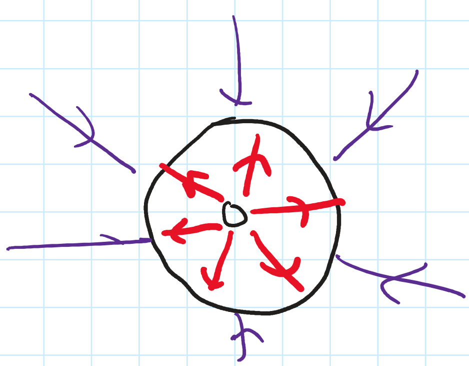
$\mathrm{Id}_{\{x_0\}} \circ \varphi = \varphi \sim \mathrm{Id}_X$ by assumption

$\varphi \circ \mathrm{Id} = \varphi(x_0) = x_0$, so $\varphi \circ \mathrm{Id} = \mathrm{Id}_{\{x_0\}}$

Other direction → exercise ⊠

Cor $\mathbb{R}^n \underset{h.eq.}{\simeq} \{pt\} \underset{h.eq}{\simeq} D^n$ $\quad P$ convex set in $\mathbb{R}^n$ $\Rightarrow P \underset{hom\,eq.}{\simeq} \{pt\}$.

Ex $\mathbb{R}^2 - \{0\}$ is homotopy eq. to S^1



$$\begin{tikzcd} \mathbb{R}^2 - \{0\} \arrow[r, bend left, "\frac{x}{\|x\|}"] & S^1 \arrow[l, bend left, "\mathrm{Id}"] \end{tikzcd}$$

$$\begin{tikzcd} S^1 \arrow[r, "\mathrm{Id}"] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathbb{R}^2 - \{0\} \arrow[r, "\frac{x}{\|x\|}"] & S^1 \end{tikzcd}$$

this comp. is identity.

Other composition:

$\mathbb{R}^2 - \{0\} \xrightarrow{\frac{x}{\|x\|}} S^1 \hookrightarrow \mathbb{R}^2 - \{0\}$ $\quad$ Need to move:

$$\begin{tikzcd} \mathbb{R}^2-\{0\} \arrow[r, "\frac{x}{\|x\|}"] \arrow[rr, bend right, "x/\|x\|"'] & S^1 \arrow[r, hook] & \mathbb{R}^2-\{0\} \end{tikzcd}$$

<u>Need to prove</u>:

$$\frac{x}{\|x\|} \sim \mathrm{Id}_{\mathbb{R}^2\setminus\{0\}}$$

<u>Homotopy</u>: $f_t(x) = tx + (1-t)\frac{x}{\|x\|}$

$$f_0(x) = \frac{x}{\|x\|} \qquad f_1(x) = x$$

Need to check that f_t is well defined, that is, $\mathbb{R}^2-\{0\} \to \mathbb{R}^2-\{0\}$,

$$tx + (1-t)\frac{x}{\|x\|} \neq 0$$

Never allowed to cross 0!

$$\Big\| \qquad \underbrace{x}_{\text{nonzero vector}}\Big(\underbrace{t}_{\geq 0} + \underbrace{\frac{(1-t)}{\|x\|}}_{\geq 0}\Big)$$

here we use $0 \leq t \leq 1$

<u>Similarly</u> $\mathbb{R}^n\setminus\{0\}$ is homotopy equivalent to S^{n-1}! (same proof).

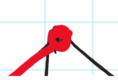
<u>Fact</u> (HW2) Homotopy equivalence of spaces is an equivalence relation.

$$\begin{tikzcd} X \arrow[r, bend left, "f_1"] & Y \arrow[l, bend left, "g_1"] \arrow[r, bend left, "f_2"] & Z \arrow[l, bend left, "g_2"] \end{tikzcd}$$

[use stuff from lec 3]

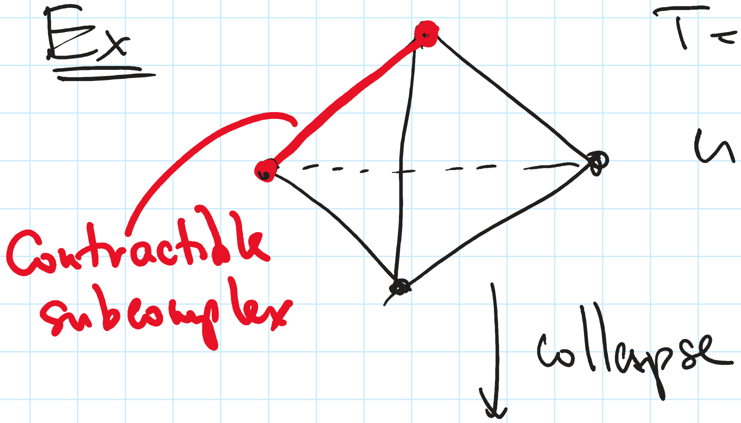
<u>Thm</u> (Hatcher 0.17) Suppose X = cell complex, A = <u>contractible</u> subcomplex. Then X is homotopy equivalent to X/A. ("collapsing a contractible subcomplex")

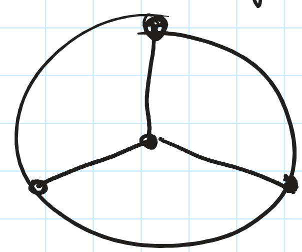
<u>Proof</u>: next time, uses homotopy extension thm.

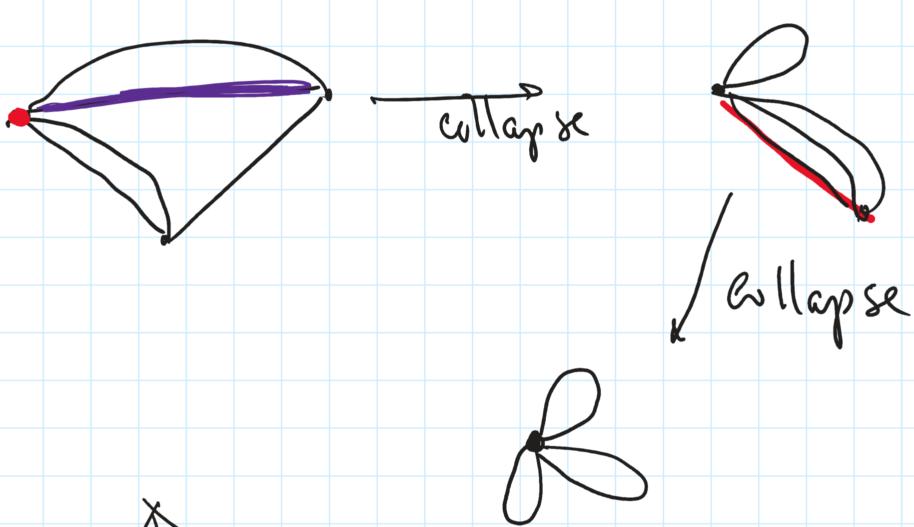
<u>Ex</u>  T = 1-skeleton of a tetrahedron

Ex

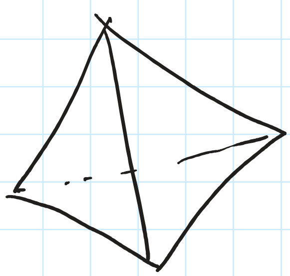
T = 1-skeleton of a tetrahedron is homeomorphic to



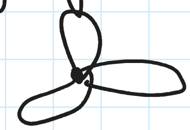




Therefore,



is homotopy equivalent to



HW2: prove any connected graph is homotopy equiv. to a graph with 1 vertex.