Lecture 5(1/20)

Friday, January 20, 2023 10:03 AM

$$\mathbb{C}P^n \neq \mathbb{R}P^{2n}$$

$$\mathbb{C}P^1 = S^2 = \mathbb{C} \cup \{\infty\} \neq \mathbb{R}P^2$$

$[z_1 : z_2]$ $\quad z_2 \neq 0 \Rightarrow z_2 = 1 \quad [z_1 : z_2] \sim [\frac{z_1}{z_2} : 1]$ $\quad \mathbb{C}$

$\lambda \in \mathbb{C}$ $[\lambda z_1 : \lambda z_2]$ $\quad z_2 = 0 \Rightarrow z_1 \neq 0 \quad [z_1 : 0] \sim [1, 0] \quad \{\infty\}$

$\lambda \neq 0$

$\mathbb{C}P^n = [z_1 : \ldots : z_{n+1}]$ $\quad z_i \in \mathbb{C}$, not all $= 0$

$\quad [\lambda z_1 : \ldots : \lambda z_{n+1}]$ $\quad \lambda \neq 0 \quad \lambda \in \mathbb{C}$

$\mathbb{R}P^{2n} = [x_1 : \ldots : x_{2n+1}]$ $\quad x_i \in \mathbb{R}$ not all $= 0$

$\quad [\lambda x_1 : \ldots : \lambda x_{2n+1}]$ $\quad \lambda \neq 0 \quad \underline{\underline{\lambda \in \mathbb{R}}}$

Def $X \supset Y$ We say that Y is a deformation retract of X if there is a homotopy $f_t : X \to X$, $f_0 = \mathrm{Id}_X$, $f_1(X) = Y$, $f_t|_Y = \mathrm{Id}_Y$ for all t.

($\Leftrightarrow$ all points of Y are fixed, X "shrinks" continuously onto Y)

Ex (see last lecture) $\mathbb{R}^2 \setminus \{0\}$ deformation retracts onto S^1

Fact If Y is a deformation retract of X then Y is homotopy equivalent to X.

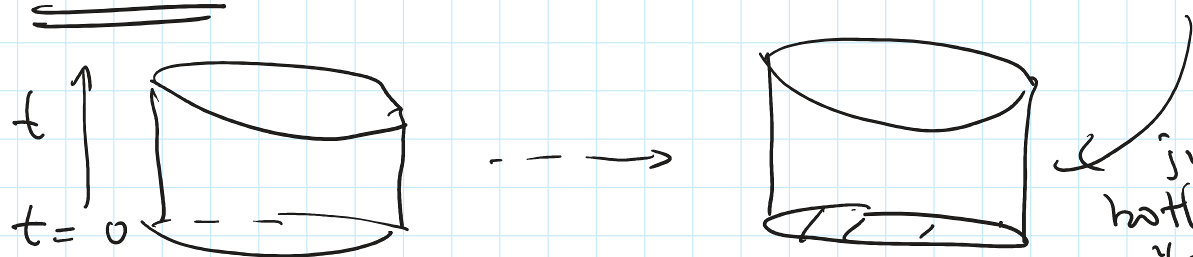
Pf f_1 $\mathrm{Id} \circ f_1 = f_1 \ldots = \mathrm{Id}$

Pf:
$$\begin{tikzcd} X \arrow[r, bend left, "f_1"] & Y \arrow[l, bend left, "Id"] \end{tikzcd}$$

$Id \circ f_1 = f_1 \sim f_0 = Id_X$

$f_1 \circ Id = f_1|_Y = Id_Y$ ☒

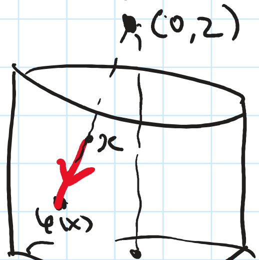
Lemma 1 $D^n \times [0,1]$ deformation retracts onto $D^n \times \{0\} \cup \partial D^n \times [0,1]$



$D^n \times [0,1]$

cup = just the bottom + "sides"

Proof



- $\varphi(x) \in D^n \times \{0\} \cup \partial D^n \times [0,1]$
- $\varphi(x)$, x and $(0,2)$ are on same line.

$\varphi_t(x) = t \cdot \varphi(x) + (1-t)x$

$t=0 \quad \varphi_0(x) = x \qquad t=1 \quad \varphi_1(x) = \varphi(x)$

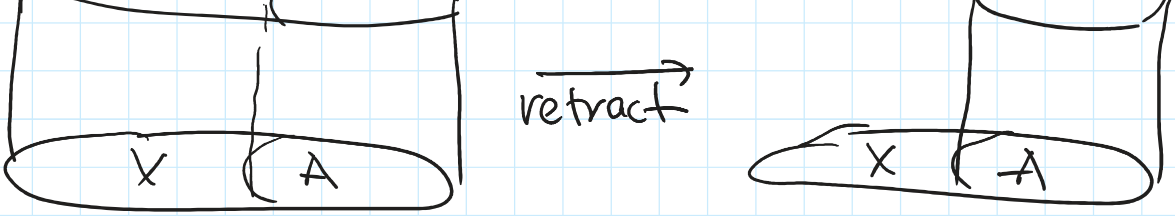
$D^n \times [0,1]$ convex $\Rightarrow \varphi_t(x) \in D^n \times [0,1]$ for all t

If $x \in D^n \times \{0\} \cup \partial D^n \times [0,1] \Rightarrow \varphi(x) = x = \varphi_t(x)$ for all t.

So this is a deformation retract. ☒

Lemma 2 X = cell complex $\supset A$ = subcomplex (closed union of cells)

Then $X \times [0,1]$ deformation retracts onto $X \times \{0\} \cup A \times [0,1]$



Proof: Lemma 1 + induction on dim of cells

$$X^n \times [0,1] \xrightarrow{\text{apply Lemma 1 to } n\text{-cells not in } A} X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times [0,1]$$

$X^n = X^{n-1} \cup$ some n-cells

could be in A or not in A

By induction assumption, we proved that:

$X^{n-1} \times [0,1]$ retracts onto $X^{n-1} \times \{0\} \cup A^{n-1} \times [0,1]$

So we get retractions

$$X^n \times [0,1] \longrightarrow X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times [0,1] \longrightarrow X^n \times \{0\} \cup A^n \times [0,1]$$

∎

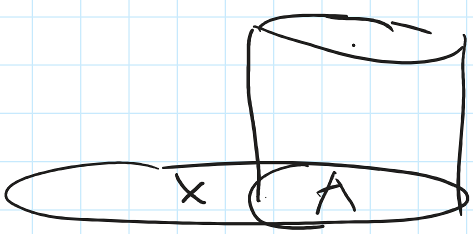
Thm (Homotopy Extension) X = cell complex, A = subcomplex

$f_0 : X \to Y$ $\quad f_0|_A = g_0$ $\quad$ Given a homotopy $g_t : A \to Y$,

can always extend it to a homotopy $f_t : X \to Y$

such that $f_t|_A = g_t$.

Proof: We are given $f_0 : X \to Y$ $\qquad G : A \times [0,1] \to Y$



Can combine them to a map

$$\begin{array}{ccc} X \times \{0\} & \cup & A \times [0,1] \longrightarrow Y \\ \downarrow f_0 & & \downarrow G \\ Y & & Y \end{array}$$

Agree on $X \times \{0\} \cap A \times [0,1] = A \times \{0\}$

$\Rightarrow$ continuous

$\Rightarrow$ continuous

$$X\times[0,1] \xrightarrow[\text{retraction from lemma2}]{} X\times\{0\}\cup A\times[0,1] \longrightarrow Y$$

This composition is a homotopy f_t. ∎

Thm X cell complex, A = contractible subcomplex

$\Rightarrow X$ is homotopy equiv to X/A.

Proof A contractible $\Rightarrow Id_A \sim \{a_0\}$ constant map

$g_t : A \to A \qquad g_0 = Id_A \qquad g_1(a) = a_0$ for all a

Use Homotopy Extension Thm to find $f_t : X \to X$

such that $f_0 = Id_X \quad f_t|_A = g_t$.

$$f_1|_A = g_1 = \{a_0\}$$

$$\begin{tikzcd} X \arrow[r, bend left, "q"] & X/A \arrow[l, bend left, "f_1"] \end{tikzcd}$$

q = projection to a quotient

f_1 collapses A to one point $\Rightarrow$ defines map $X/A \to X$

$f_1 \circ q = f_1 \sim f_0 = Id_X \quad \ldots$

So X is homotopy equivalent to X/A. ∎
