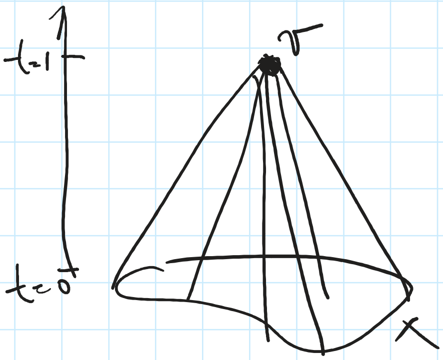



**Cones** $X$ top. space

$$CX = \mathrm{Cone}(X) = X \times [0,1] \Big/ {\scriptstyle (x,1) \sim (x',1) \text{ for all } x, x'}$$

$$= X \times [0,1] \Big/ X \times \{1\}$$

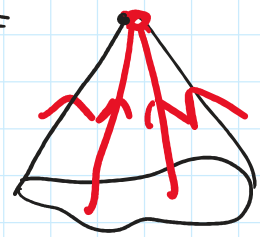
**Lemma** For any $X$, $CX$ is contractible.

**Proof**: $\varphi_0 = \mathrm{Id}_{CX}$ $\varphi_1 = \{v\}$ where $v = [(x,1)]$

constant map

$$\varphi_s((x,t)) = \text{"}(1-s)(x,t) + s(x,1)\text{"} =$$

$$= (x, (1-s)t + s)$$



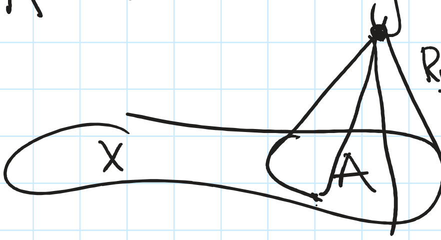
At $s=0$, $\varphi_0(x,t) = (x,t)$

$s=1$, $\varphi_1(x,t) = (x,1) \sim v$ for all $x$

So $\varphi_0$ and $\varphi_1$ are homotopic, and $CX$ is contractible. ∎

---

**Thm** Suppose $X$ is a cell complex, $A$ subcomplex. Then $X/A$ is homotopy equivalent to $X \cup_A CA$.




Rmk Formally,

$$X \cup_A CA = X \cup A \times [0,1] / \sim$$

$a \sim (a,0)$, $(a,1) \sim (a',1)$ $a \in A$ for all $a' \in A$.

vertex of cone

attach the bottom of CA to X

**Cor** When we work with top. spaces up to homotopy equivalence, we can always replace $X/A$ with quotient topology by an actual space $X \cup_A CA$.

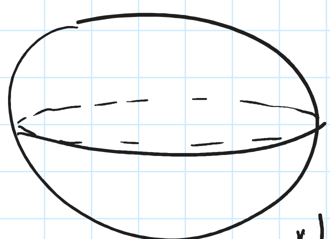
**Proof** $X/A$ is homeomorphic to $(X \cup_A CA)/CA$

By Lemma, $CA$ is contractible, therefore
$X \cup_A CA$ is homotopy equivalent to $(X \cup_A CA)/CA$.

(We proved last time that collapsing a contractible subset does not change homotopy type). ∎

---

**Ex** $S^2$

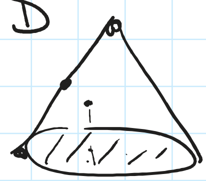


$S^1$

$S^2/S^1 = ?$

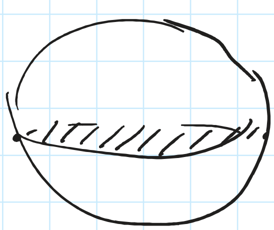
By thm, $S^2/S^1 \underbrace{\simeq}_{\text{homotopy eq.}} S^2 \cup_{S^1} CS^1$

Now $CS^1 \simeq D^2$

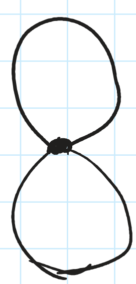


$\downarrow \pi$

Therefore $S^2 \cup_{S^1} CS^1$ is homotopy equivalent to



collapse disk.



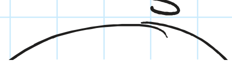
$S^2$

$S^2$

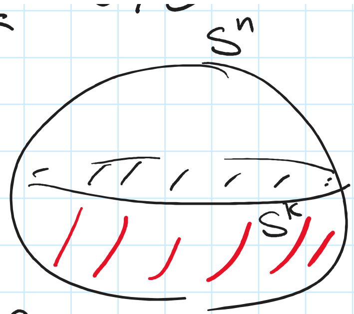
"bouquet of two $S^2$" $= S^2 \cup_{pt} S^2$.

**Ex** $S^n/S^k$ where $n > k$. where $S^k$ is standardly embedded into $S^n$

$S^n$



embedded into $S^n$




$$S^n/S^k = S^n \cup_{S^k} CS^k = S^n \cup_{S^k} D^{k+1}$$

since $CS^k = D^{k+1}$

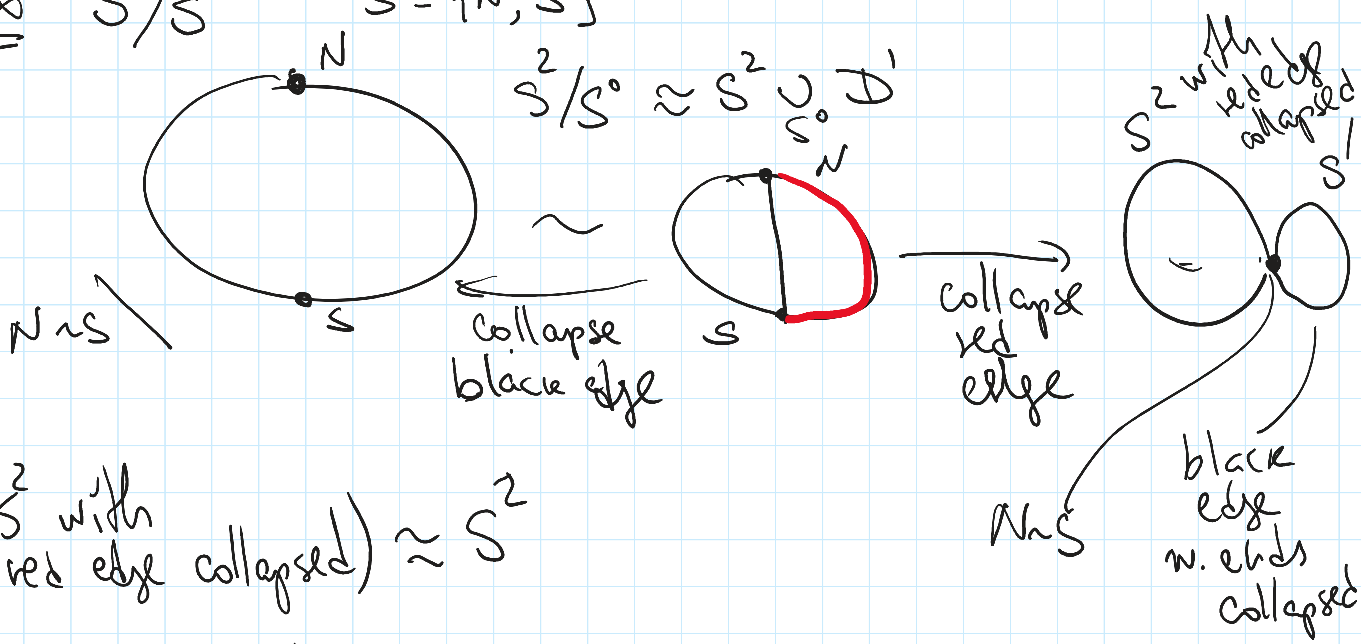
Observe (we'll prove it later) that $S^k$ bounds a disk $D^{k+1}$ in $S^n$. Then

$$S^n \cup_{S^k} D^{k+1} \simeq (\text{homotopy eq.}) \simeq \left(S^n \cup_{S^k} D^{k+1}\right) \big/ D^{k+1} =$$

$$= \left(S^n / D^{k+1}\right) \cup_{pt} \left(D^{k+1}/S^k\right) \simeq \boxed{S^n \cup_{pt} S^{k+1}}$$

<u>Ex</u> $S^2/S^0$ $\quad S^0 = \{N, S\}$

$$S^2/S^0 \simeq S^2 \cup_{S^0} D^1$$




$\left(S^2 \text{ with red edge collapsed}\right) \simeq S^2$

<u>Rmk</u> $S^k \subset S^n$ embedded in standard way $(k < n)$ bounds a disk in $S^n$. Indeed,

$$S^k \subset S^{k+1} \subset S^n$$

$S^k$ is an equatorial sphere in $S^{k+1}$ $\Rightarrow$ bounds a disk in $S^{k+1} \subset S^n$.

---

<u>Thm</u> Suppose $f, g : A \to Y$ are homotopic continuous maps. Then the results of gluing, $A$ subcomplex in $X$.

$$X \cup_f Y = X \cup Y / (a \sim f(a) \text{ for all } a \in A)$$

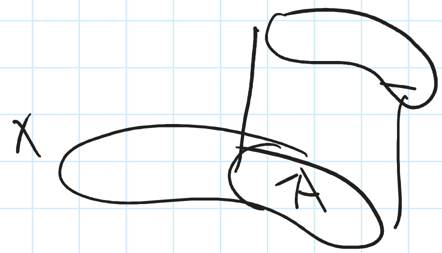
gluing $X$ to $Y$ along $A$ with attaching map $f$

$$X \cup_g Y = X \cup Y / (a \sim g(a) \text{ for all } a \in A)$$


$Y$ Then $X \cup_f Y$ and $X \cup_g Y$ are homotopy equivalent.

$a \sim f(a)$.

<u>Proof</u> $F : A \times [0,1] \to Y$ such that $F(a,0) = f(a)$
$F(a,1) = g(a)$.




(*) $(X \times \{0\} \cup A \times [0,1]) \cup_F Y$ is homeomorphic to $X \cup_f Y$

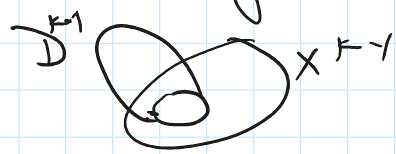
(**) $(X \times \{1\} \cup A \times [0,1]) \cup_F Y$ is homeomorphic to $X \cup_g Y$.


But we proved a lemma in last lecture that $X \times [0,1]$ deformation retracts onto $X \times \{0\} \cup A \times [0,1]$ or $X \times \{1\} \cup A \times [0,1]$

Therefore both (*) and (**) are homotopy equivalent to $X \times [0,1] \cup_F Y \Rightarrow$ homotopy equivalent to each other.

__Very Important Corollary__ Suppose that we attach a cell $D^k$ to $(k-1)$ skeleton $X^{k-1}$ along two attaching maps $f: \partial D^k \longrightarrow X^{k-1}$

$g: \partial D^k \longrightarrow X^{k-1}$




If $f$ is homotopic to $g$ then the resulting spaces $D^k \cup_f X^{k-1}$ and $D^k \cup_g X^{k-1}$ are homotopy equivalent.

In other words, to understand cell complexes up to homotopy equivalence, we need to understand attaching maps $\partial D^k = S^{k-1} \longrightarrow X^{k-1}$ up to homotopy.

In this course, we will study maps $S^{k-1} \to X$, for different $X$ (fundamental group, higher homotopy groups)