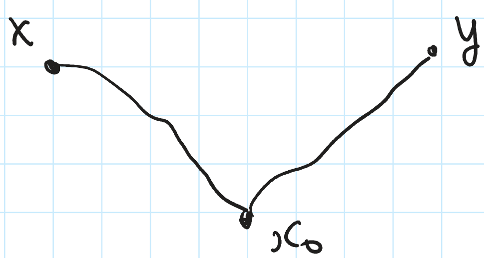
(1) $X$ contractible $\Rightarrow X$ path connected

Contractible by def: $\underset{\substack{\parallel \\ f}}{Id_X} \sim \underset{\substack{\parallel \\ g}}{\text{constant}}$ map $\{x_0\}$

$f(x)=x \qquad g(x)=x_0$ for all $x$

Homotopy: $F(x,t)$ such that $F(x,0)=f(x)=x$

$F(x,1)=g(x)=x_0$




Choose two points $x,y \in X$
Need to prove $x,y$ connected by a path.

$$\gamma(t) = \begin{cases} F(x,2t), & 0 \le t \le \frac{1}{2} \\ F(y, 2-2t), & \frac{1}{2} \le t \le 1 \end{cases}$$

$t=0$ : $F(x,0)=x$

$t=\frac{1}{2}$ : $F(x,1)=x_0=F(y,1) \Rightarrow$ continuous at $t=\frac{1}{2}$

$t=1$ : $F(y,0)=y$.

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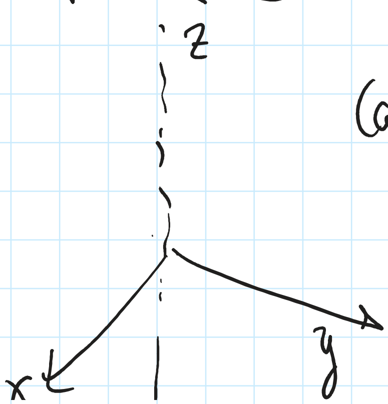
(2) $\mathbb{R}^3 \setminus Z \qquad Z=$ z-axis $\quad (0,0,z) \quad z \in \mathbb{R}$




(a) Prove $\mathbb{R}^3 \setminus Z \underset{\text{homeo}}{\simeq} \mathbb{R}^2 \times S^1$

Cylindrical coordinates:

$(x,y,z) \xrightarrow{\varphi} (r, \theta, z)$

$r = \sqrt{x^2+y^2}$

$0 \le \theta \le 2\pi \quad \in S^1 \rightsquigarrow \left(\frac{x}{r}, \frac{y}{r}\right) \in S^1$

Well defined since $r \neq 0$ $r > 0$.

<u>Issue</u>: $r > 0$!

$\varphi : \mathbb{R}^3 \setminus z \longrightarrow \mathbb{R}_{>0} \times S^1 \times \mathbb{R}$

bijection

Need to compose this with a bijection $\mathbb{R}_{>0} \longrightarrow \mathbb{R}$
(homeomorphism)

For example, $\ln(x) : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$

inverse $e^x : \mathbb{R} \longrightarrow \mathbb{R}_{>0}$

Total: $(x, y, z) \longrightarrow \left(\underset{\mathbb{R}}{\ln(r)}, \underset{\mathbb{R}}{z}, \underset{S^1}{\left(\frac{x}{r}, \frac{y}{r}\right)}\right)$

(b) $\underset{\text{contractible}}{\mathbb{R}^2} \times S^1 \underset{\text{homotopy eq.}}{\sim} \{0\} \times S^1 = S^1$

$$\begin{tikzcd} (t_1, t_2, \varphi) \arrow[r, shift left, "\pi"] & (0, 0, \varphi) \arrow[l, shift left, "id"] \end{tikzcd}$$

$\pi \circ id = Id_{S^1}$ $\qquad id \circ \pi (t_1, t_2, \varphi) = (0, 0, \varphi)$

Need to prove it's homotopic to $Id_{\mathbb{R}^2 \times S^1}$

$$F_s(t_1, t_2, \varphi) = (st_1, st_2, \varphi)$$

$s = 1 \quad (t_1, t_2, \varphi)$

$s = 0 \quad (0, 0, \varphi)$.

$s = 0 \quad (0, 0, \varphi)$.

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Rmk If $X$ is homotopy equivalent to $Y$ then $X \times Z$ is homotopy equivalent to $Y \times Z$. for all $Z$

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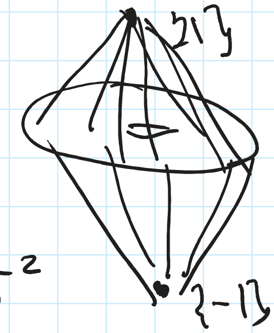
(3) $S(T^2) = T^2 \times [-1,1]/\sim$

$\{1\}\ (p, 1) \sim (q, 1)$ for all

$\{-1\}\ (p, -1) \sim (q, -1)$ $p, q \in T^2$

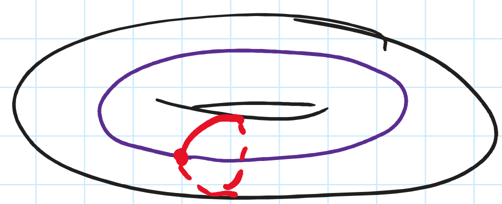



For example, pick cell decomp of $T^2$ with

one 0-cell

two 1-cells

one 2-cell



$[-1,-1]$ has cell decomp with 0-cells $\{-1\}$, $\{1\}$ and 1-cell

$S(T^2)$: $\{1\}$, $\{-1\}$, (interval) $\times$ (cells in $T^2$)

0-cells,

one 1-cell

two 2-cells

one 3-cell

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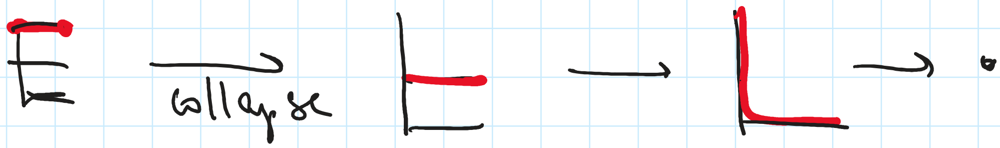
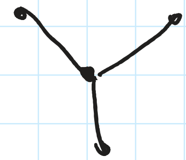
(4) F L U T E

(a) All of these are contractible



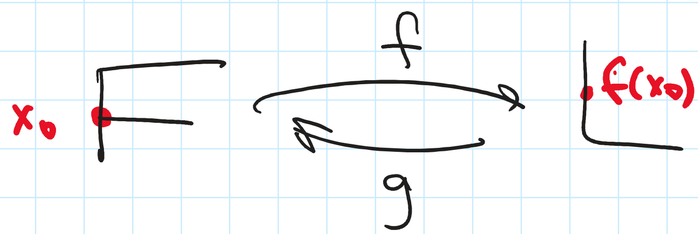

(b) $F \underset{\text{homeo}}{\simeq} T \underset{\text{homeo}}{\simeq} E$



$L \underset{\text{homeo}}{\simeq} U \simeq$ 

Why these are not homeomorphic to each other, say $F$ is not homeo to $L$?

Idea: by contradiction. Suppose there's a homeo




$x_0$ = trivalent point in $F$

$\longleftrightarrow f(x_0)$ in $L$

$F - \{0\}$ has three connected components

but $L - \{a\}$ has one or two connected component for any point $a \in L$.

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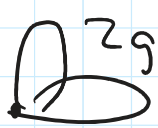
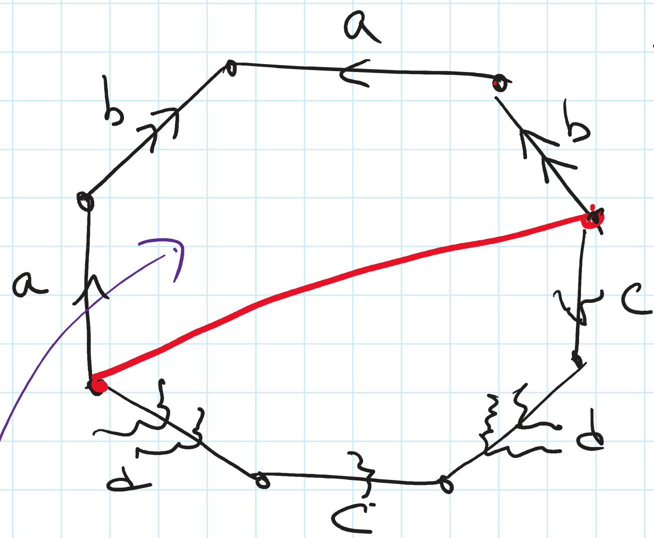
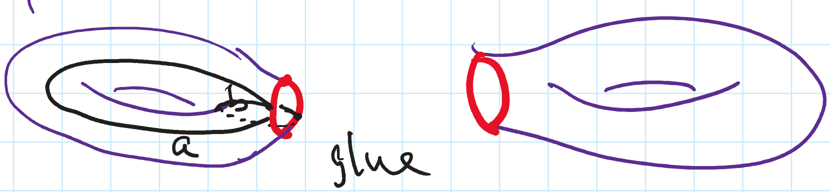
⑤* $\Sigma_g$ = genus $g$ surface

Prove $\Sigma_g \setminus \{pt\} \simeq$ graph  1 vertex, $2g$ edges

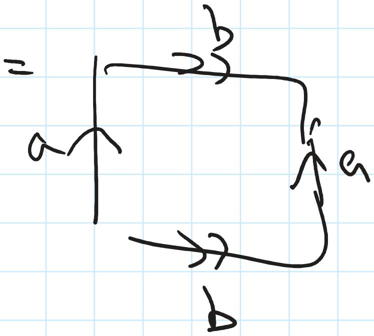

Idea: glue $\Sigma$ from a

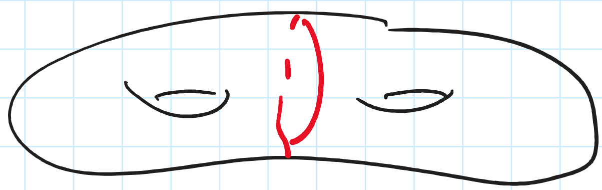



**Idea:** glue $\Sigma_g$ from a polygon with $4g$ sides

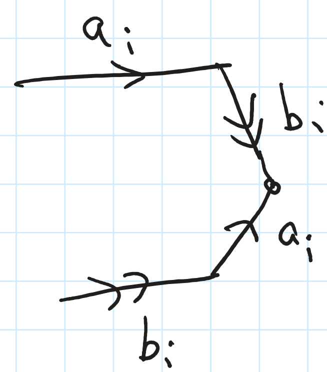



**Proof:** $T^2 =$



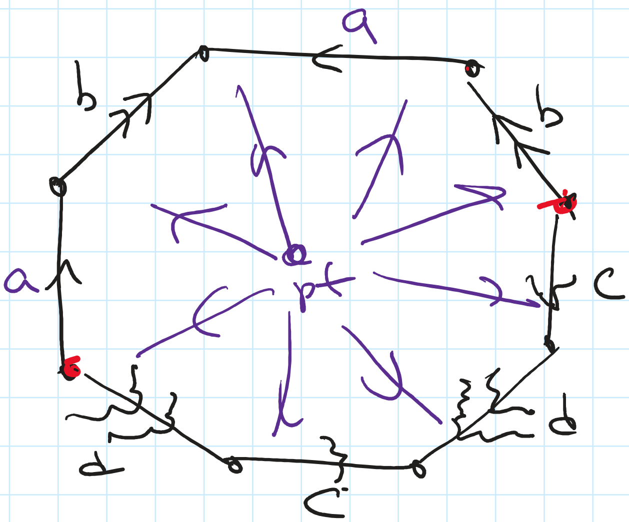



In general, $4g$ sides




$\leftarrow g$ of these

$1 \le i \le g$

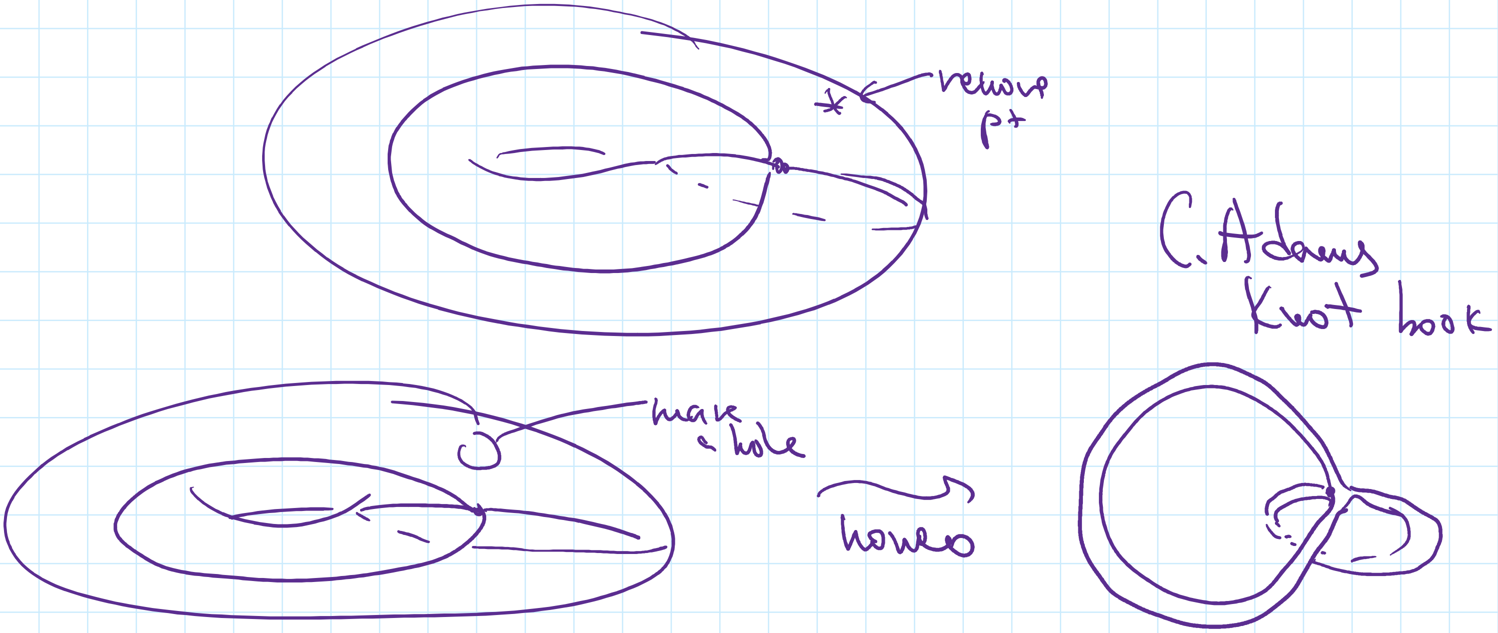



$4g$-gon $- \{pt\}$ can be retracted to the boundary.

can choose pt to be the center

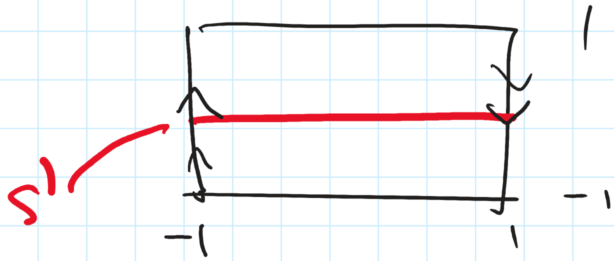
$\longrightarrow$ $4g$ edges glued $\Rightarrow$ $2g$ edges & 1 vertex

**Ex** $T^2 \setminus \{pt\}$




C. Adams Knot book

## HW2 #1 Mobius band $\sim S^1$




$$[-1,1]\times[-1,1]/\sim$$
$$(-1,y)\sim(1,-y)$$

$S^1$ since $(-1,0)\sim(1,0)$

Retraction: $F_t(x,y) = (x, yt)$

$t=0$ $(x,0)$ $\quad$ $t=1$ $(x,y)$

Note: Need to check that the homotopy is well defined!

$$F_t(-1,y) = (-1,yt)$$
$\wr$ $\quad$ $\wr$ equivalent for all $t$!
$$F_t(1,-y) = (1,-yt)$$