Lecture 8 (2/1)

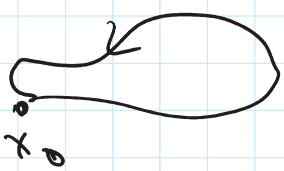
Wednesday, February 1, 2023 9:52 AM

Fundamental group

X = top. space fix a basepoint $x_0 \in X$

Def A loop in X is a continuous map

$\gamma : [0,1]_s \to X$ such that $\gamma(0) = \gamma(1) = x_0$



Def $\pi_1(X)$ = fundamental group of X

is

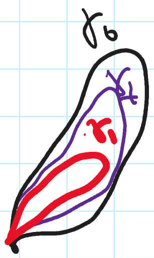
$$\pi_1(X) = \left\{ \begin{array}{c} \text{all loops} \\ \text{starting and} \\ \text{ending at } x_0 \end{array} \right\} \Big/ \sim$$

$\pi_1(X, x_0)$

$\sim$ homotopy of loops fixing the endpoints.

$\gamma_0 : [0,1] \to X \qquad \gamma_1 : [0,1] \to X$

$\gamma_0 \sim \gamma_1$ if there is a homotopy



$\gamma_t : [0,1] \longrightarrow X$ such that

at $t=0$ get γ_0, $t=1$ get γ_1, and

$\gamma_t(0) = \gamma_t(1) = x_0$ for all t.

This is a group! Composition of loops:

$\gamma_1 * \gamma_2$ = new loop

$$\gamma(s) = \begin{cases} \gamma_1(2s), & 0 \le s \le \frac{1}{2} \\ \gamma_2(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$\left\{ \gamma_2(2s-1), \ \tfrac{1}{2} \le s \le 1 \right.$$

At $s = 1/2$, we get $\gamma(\tfrac{1}{2}) = \gamma_1(1) = x_0 = \gamma_2(0)$
so this is continuous.

$s=0 \quad \gamma(0) = \gamma_1(0) = x_0 \qquad s=1 \quad \gamma(1) = \gamma_2(1) = x_0.$

<u>Thm</u> (a) The composition is well defined:
$\gamma_1 \sim \gamma_1'$, then $\gamma_1 * \gamma_2 \sim \gamma_1' * \gamma_2$

(b) $\pi_1(X)$ is a group
- identity
- associativity
- inverse

s = loop parameter
t = homotopy

<u>Proof</u> (a) $\gamma_1 \sim \gamma_1'$ then there is a homotopy
$\gamma : [0,1] \to X \qquad \gamma^{(0)} = \gamma_1 \qquad \gamma^{(1)} = \gamma_1'$

$$\begin{matrix} \gamma_1(2s), & s \le \tfrac{1}{2} \\ \gamma_2(2s-1), & s \ge \tfrac{1}{2} \end{matrix} \qquad \tilde{\gamma}^{(t)} = \begin{cases} \gamma^{(t)}(2s), & s \le \tfrac{1}{2} \\ \gamma_2(2s-1), & s \ge \tfrac{1}{2} \end{cases}$$

Since $\gamma^{(t)}(1) = x_0$ for <u>all t</u>, this is continuous
for all t.

$\tilde{\gamma}^{(t)} = \gamma^{(t)} * \gamma_2$

$\tilde{\gamma}^{(0)} = \gamma^{(0)} * \gamma_2 = \gamma_1 * \gamma_2$

$$\tilde{\gamma}^{(1)} = \gamma^{(1)} * \gamma_2 = \gamma_1' * \gamma_2 .$$

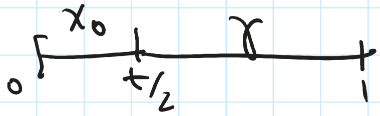
(b) <u>Identity</u> : $e(s) = x_0 \quad 0 \le s \le 1$

constant loop.

$$e * \gamma(s) = \begin{cases} x_0 , & 0 \le s \le \frac{1}{2} \\ \gamma(2s-1) , & \frac{1}{2} \le s \le 1 \end{cases}$$

This is <u>NOT</u> equal to γ but it is homotopic to γ.

$$\Gamma_t(s) = \begin{cases} x_0 , & 0 \le s \le \frac{t}{2} \\ \gamma\left(\frac{s - t/2}{1 - t/2}\right) , & \frac{t}{2} \le s \le 1 \end{cases}$$



- Continuous in s:

$$\Gamma_t\left(s = t/2\right) = x_0 = \gamma(0)$$

- loop for all t: $\Gamma_t(0) = x_0 \qquad \Gamma_t(1) = \gamma\left(\frac{1 - t/2}{1 - t/2}\right) =$

$$= \gamma(1) = x_0$$

- At $t = 0$ get $\underline{\gamma(s)} = \Gamma_0(s)$

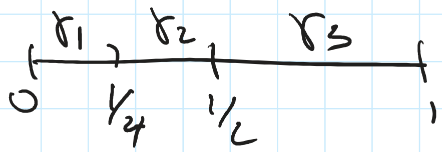
At $t = 1$ get $\underline{e * \gamma} = \Gamma_1(s)$

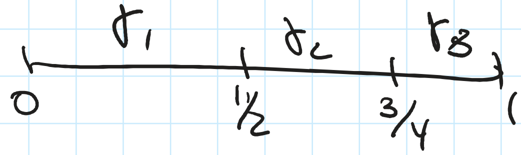
<u>Conclusion</u> $e * \gamma \sim \gamma$ for all γ, so e is the identity.

- <u>Associativity</u> $(\gamma_1 * \gamma_2) * \gamma_3 \sim \gamma_1 * (\gamma_2 * \gamma_3)$

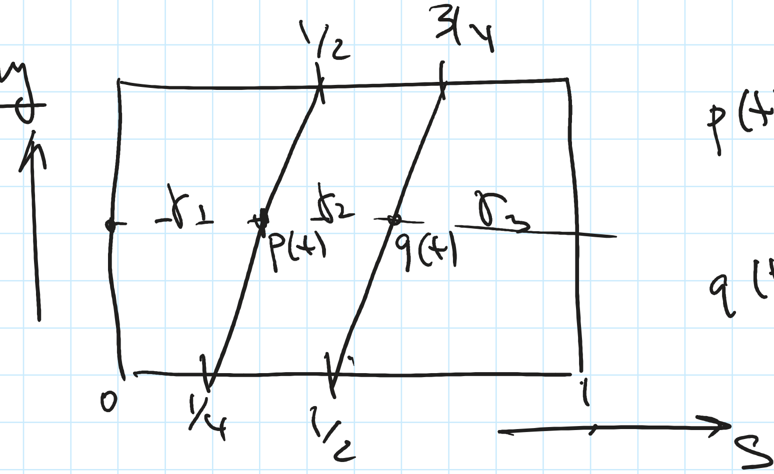
$$\begin{cases} \gamma_1(4s) , & 0 \le s \le 1/4 \\ \gamma_2(4s-1) , & \frac{1}{4} \le s \le 1/2 \end{cases} \qquad \begin{cases} \gamma_1(2s) , & 0 \le s \le \frac{1}{2} \\ \gamma_2(4s-2) , & \frac{1}{2} \le s \le 3/4 \end{cases}$$

$$\left\{\begin{array}{l}\gamma_1(4s),\ 0\le s\le 1/4\\ \gamma_2(4s-1),\ \tfrac{1}{2}\le s\le 1/2\\ \gamma_3(2s-1),\ \tfrac{1}{2}\le s\le 1\end{array}\right. \qquad \left\{\begin{array}{l}\gamma_1(2s),\ 0\le s\le \tfrac{1}{2}\\ \gamma_2(4s-2),\ \tfrac{1}{2}\le s\le 3/4\\ \gamma_3(4s-3),\ \tfrac{3}{4}\le s\le 1\end{array}\right.$$





Homotopy

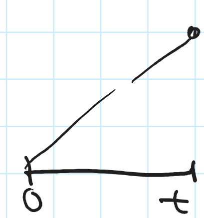


$$p(t) = t\cdot \tfrac{1}{2} + (1-t)\cdot \tfrac{1}{4}$$
$$= \tfrac{1}{4} + \tfrac{1}{4}t$$
$$q(t) = \tfrac{1}{2} + \tfrac{1}{4}t$$

$$\Gamma_t(s) = \left\{\begin{array}{l}\gamma_1\left(\dfrac{s}{p(t)}\right),\ 0\le s\le p(t)\\ \gamma_2\left(\dfrac{s-p(t)}{q(t)-p(t)}\right)\ p(t)\le s\le q(t)\\ \gamma_3\left(\dfrac{s-q(t)}{1-q(t)}\right),\ q(t)\le s\le 1\end{array}\right.$$

Inverse $\quad \gamma^{-1}(s) = \gamma(1-s)$

$$\gamma*\gamma^{-1}(s) = \left\{\begin{array}{l}\gamma(2s),\ 0\le s\le 1/2\\ \gamma(1-(2s-1)) = \gamma(2-2s),\ \tfrac{1}{2}\le s\le 1\end{array}\right.$$



$$\Gamma_t(s) = \left\{\begin{array}{l}\gamma(2ts),\ 0\le s\le 1/2\\ \ldots,\ 1/2\le s\le 1\end{array}\right.$$

$$\Gamma_t(s) = \begin{cases} \gamma(\ \ \ ), & \ \ \le \tfrac{1}{2} \\ \gamma((1-s)\cdot 2t), & \tfrac{1}{2} \le s \le 1 \end{cases}$$

0 t

linear fn

$f(1/2) = t \quad f(1) = 0$

$t = 0 \quad e = \Gamma_0(s)$

$t = 1 \quad \gamma * \gamma^{-1} = \Gamma_1(s)$

Conclusion: $\gamma * \gamma^{-1} \sim e$.

<u>Facts</u> about $\pi_1(X)$: (to be proved later)

- X contractible $\Rightarrow \pi_1(X) = \{e\}$
- $\pi_1(S^1) = \mathbb{Z}$! So S^1 is not contractible
- $\pi_1(X)$ does not need to be abelian
- $f : X \longrightarrow Y$ continuous
  $f_* : \pi_1(X) \longrightarrow \pi_1(Y)$ group homomorphism.
- X = CW complex $\Rightarrow \pi_1(X)$ can be described explicitly by generators and relations.
- If X is homotopy eq. to Y then
  $\pi_1(X) \simeq \pi_1(Y)$