2/3

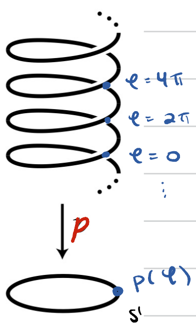
Theorem $\boxed{\pi_1(S^1) = \mathbb{Z}}$

$$\pi_1(S^1) = \{ \text{loops } \gamma : [0,1] \to S^1 \mid \gamma(0) = \gamma(1) = (1,0) \} / \sim$$

Want a degree map, $\deg : \{\text{loops in } S^1\} \to \mathbb{Z}$

such that

- Two loops of different degrees are <u>not</u> homotopic
- Two loops of the same degree are homotopic
- $\deg(\gamma_1 * \gamma_2) = \deg(\gamma_1) + \deg(\gamma_2)$



Consider the map

$$p : \mathbb{R} \longrightarrow S^1$$
$$\varphi \longmapsto (\cos\varphi, \sin\varphi) \quad \text{in } \mathbb{R}^2 \approx e^{i\varphi}$$

infinite # of points in $\mathbb{R} \longleftrightarrow$ point in S^1

$$\varphi + 2\pi k \longmapsto p(\varphi)$$

Key Lemma

(1) Given a loop $\gamma : [0,1] \to S^1$ such that $\gamma(0) = \gamma(1) = (1,0)$ there is a unique lift $f_\gamma : [0,1] \to \mathbb{R}$ of γ such that

$$\gamma(s) = p \circ f_\gamma(s)$$
$$= (\cos f_\gamma(s), \sin f_\gamma(s))$$
$$= e^{i f_\gamma(s)}$$

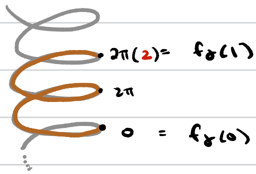
$$\begin{tikzcd} {[0,1]} \arrow[r, "\gamma"] \arrow[dr, dashed, "f_\gamma"'] & S^1 \arrow[d, phantom, "\circlearrowleft"] \\ & \mathbb{R} \arrow[u, "p"'] \end{tikzcd}$$

and $f_\gamma(0) = 0$

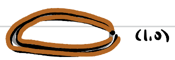
$(\cos 2\pi K, \sin 2\pi K)$

γ is a loop $\iff \gamma(1) = (1,0)$

$\iff f_\gamma(1) = 2\pi K$ for some K



Def the degree of a loop γ is K defined above



loop around twice

degree $\gamma = 2$

Lemma cont.

(2) Given a homotopy of loops γ_t,

$\exists$ homotopy of lifts f_{γ_t} continuous in t

Lemma $\Rightarrow$ Theorem

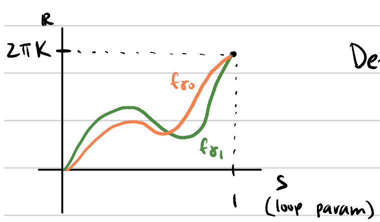
Proof: If γ_t is a homotopy of loops, $f_{\gamma_t} : [0,1] \to \mathbb{R}$ is a homotopy of lifts and $f_{\gamma_t}(1) = 2\pi K(t)$

so $K(t) : [0,1]_t \to \mathbb{Z}$ is a continuous function

hence $K(t)$ is a constant map.

Conclusion: $\gamma_0 \sim \gamma_1 \Rightarrow \deg \gamma_0 = \deg \gamma_1$

Conversely, suppose we know $\deg \gamma_0 = \deg \gamma_1 = K$



Define $f_t(s) = t f_{\gamma_0}(s) + (1-t) f_{\gamma_1}(s)$

then $f_t(0) = 0$ and $f_t(1) = 2\pi K$

Define $\gamma_t(s) = (\cos f_t(s), \sin f_t(s))$

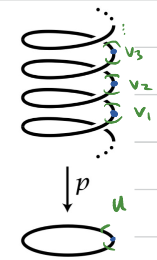
$\Rightarrow \gamma_t$ is a homotopy between γ_0, γ_1 !

so $\quad \gamma_0 \sim \gamma_1 \quad$ iff $\quad \deg \gamma_0 = \deg \gamma_1$

Note we can consider linear function $f_K(s) = 2\pi K s$

$\gamma_K(s) = (\cos(2\pi K s), \sin(2\pi K s))$ is a map of degree K

for $t < \delta$, $\gamma(t) \in \mathcal{U}$



$p^{-1}(\underbrace{\text{small interval in } S^1}_{= \mathcal{U}}) = \mathbb{Z}$ disjoint intervals in $\mathbb{R}$

Claim: $\quad p^{-1}(\mathcal{U}) = \bigsqcup_{K \in \mathbb{Z}} V_K \quad , \quad V_K$ open in $\mathbb{R}$

and each V_K homeomorphic to $\mathcal{U}$

* Example of a covering space

- Suppose $\gamma(t) : [0,1] \to S^1$ s.t. $\gamma([t,t']) \subseteq \mathcal{U}$ for some $t<t'$
  If $\gamma(t) \in \mathcal{U}$ and $t' - t < \delta$ for some δ, then $\gamma([t,t']) \in \mathcal{U}$
- For each K $\exists$ unique lift f of $\gamma([t,t'])$ contained in V_K
  $f([t,t']) \subset V_K \quad , \quad \gamma = (\cos f, \sin f)$
- Use a homeo $V_K \cong \mathcal{U}$.
  Since $f([t,t'])$ is connected, it is contained in only one V_K

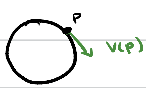
Winding Number

(# of times a loop wraps around the origin)

$\gamma : S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\} \qquad$ (loop in $\mathbb{R}^2 \setminus \{0\}$)

$\dfrac{\gamma}{\|\gamma\|} : S^1 \longrightarrow S^1$ has a winding number $\deg\left(\frac{\gamma}{\|\gamma\|}\right)$

Vector Field on $\mathbb{R}^n$



vector at every point.

$\dfrac{V}{\|V\|} : S^1 \longrightarrow S^1$