MAT 215A, Winter 2023
Homework 1

Due before 10:00 on Wednesday, January 18

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

1. Prove that the following spaces are homeomorphic to each other by constructing explicit homeomorphisms:

a) The **real projective space** $\mathbb{RP}^n$: the space of lines in $\mathbb{R}^{n+1}$ passing thought the origin.

b) The quotient $(\mathbb{R}^{n+1} \setminus \mathbf{0})/\sim$ by the equivalence relation

$$(x_1, \ldots, x_{n+1}) \sim (\lambda x_1, \ldots, \lambda x_{n+1}), \lambda \neq 0.$$

c) The quotient $S^n/\sim$ by the equivalence relation

$$(x_1, \ldots, x_{n+1}) \sim (-x_1, \ldots, -x_{n+1}).$$

d) The quotient $D^n/\sim$ by the equivalence relation which identifies the opposite points **only** at the boundary $\partial D^n = S^{n-1}$.

2. Construct a cell decomposition of $\mathbb{RP}^n$. How many cells of each dimension are there?

3. The **complex projective space** $\mathbb{CP}^n$ is the space of complex lines in $\mathbb{C}^{n+1}$ passing thought the origin.

a) Construct a cell decomposition of $\mathbb{CP}^n$. How many cells of each dimension are there?

b) Prove that $\mathbb{CP}^n$ is homeomorphic to the quotient $S^{2n+1}/\sim$ by the equivalence relation

$$(z_1, \ldots, z_{n+1}) \sim (\lambda z_1, \ldots, \lambda z_{n+1}), \ |\lambda| = 1.$$

Here

$$S^{2n+1} = \left\{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : |z_1|^2 + \ldots + |z_{n+1}|^2 = 1\right\}.$$

4. The suspension SX of a topological space X is defined as the quotient of $X \times [0, 1]$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

a) Prove that $S(D^n) = D^{n+1}$ and $S(S^n) = S^{n+1}$.

b) Given a continuous map $f : X \to Y$, define a map $Sf : SX \to SY$ and prove it's continuous. Prove that $S(f \circ g) = Sf \circ Sg$ for arbitrary composable continuous maps f, g.

c) Given a cell decomposition of an arbitrary cell complex X, construct a cell decomposition of SX.