Lecture 1 (3/29)

Monday, March 29, 2021 2:07 PM

Cohomology = "dual of homology"
precise definition later today

Similar to homology, has several advantages:

1. Cup product: cohomology of any f.g. space has multiplication, so it is a ring.
2. Poincaré duality: for smooth manifolds, homology and cohomology are related in a nontrivial way
3. De Rham cohomology: differential forms correspond to cohomology classes.
4. Obstruction theory: invariants of fiber bundles are often classified by cohomology. (ex: classify all S^1-bundles on a given space X). $E \xrightarrow{S^1} X$

on a given space X). $E \xrightarrow{s} X$

Linear algebra recap:

V = vector space over $\mathbb{K}$

V^* = dual space = space of linear functions $V \longrightarrow \mathbb{K}$.

- If $\dim V = n$ then $\dim V^* = n$

  $e_1 \dots e_n$ = basis in V $\quad e_1^* \dots e_n^*$ dual basis

  $$e_i^*(a_1 e_1 + \dots + a_n e_n) = a_i$$

- $U \xrightarrow{f} V$ linear map

  $\Rightarrow f^* : V^* \longrightarrow U^*$ dual map

  $$f^*(\alpha)(u) = \alpha(f(u))$$

  linear function on V

  vector in U

- $U \subset V$ subspace

  $\mathrm{Ann}(U) = \{\alpha \in V^* \mid \alpha(u) = 0 \text{ for all } u \in U\}$

  annihilator of U

<u>**Fact**</u> $\dim \mathrm{Ann}\, U = \dim V - \dim U$

Pf: [illegible] basis in U

Pf: $e_1 \dots e_k$ = basis in U

$e_1 \dots e_n$ = basis in V $\quad \mathrm{Ann}(U) = \mathrm{Span}(e_{k+1}^*, \dots e_n^*)$

HW1 $\mathcal{U} \xrightarrow{f} V$, prove that

(a) $\mathrm{Ann}(\underbrace{\mathrm{Im} f}_{\text{subspace of } V}) = \mathrm{Ker} f^*$

(b) $\mathrm{Ann}(\underbrace{\mathrm{Ker} f}_{\text{subspace of } U}) = \mathrm{Im} f^*$

$\mathcal{U}, V$ f. dim

$$\to C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} C_{i-1} \to \dots$$

chain complex, C_i = some vector spaces, $\partial^2 = 0$

$$\leftarrow C_{i+1}^* \xleftarrow{\delta} C_i^* \xleftarrow{\delta} C_{i-1}^* \leftarrow \dots$$

$C^{i+1} \quad C^i \quad C^{i-1}$

$\partial^* = \delta$ = dual differential

C_i = chain groups $\quad C^i = C_i^*$ cochain groups

$\alpha \in C^i$ cochain $\quad \boxed{\delta(\alpha)(x) = \alpha(\partial x)}$

$x \in C_{i+1}$

$\Rightarrow \delta(\alpha) \in C^{i+1}$

Cocycle = Ker δ $\quad$ Coboundary = Im δ.

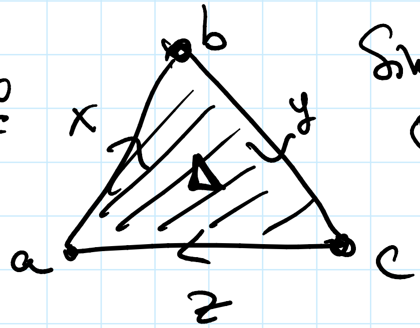
Note: If C_i are free abelian groups, we can use exactly the groups, we can use exactly the same construction $C_i^* = Hom(C_i, \mathbb{Z})$

<u>Def</u> Cohomology of $C_\bullet$ = homology of dual complex

X = top. space $\rightsquigarrow$ can define chain complex (simplicial, singular, cellular, ...)

Dualize and compute cohomology.

<u>Ex</u>



Simplicial chain complex:

$$0 \to C_2 \to C_1 \to C_0 \to 0$$
$$C_2 = \langle \Delta \rangle \quad C_1 = \langle x, y, z \rangle \quad C_0 = \langle a, b, c \rangle$$

$$\partial(\Delta) = x + y + z$$
$$\partial(x) = b - a \quad \partial(y) = c - b \quad \partial(z) = a - c$$

<u>Dual complex</u>: $0 \to C^0 \to C^1 \to C^2 \to 0$

$$C^0 = \{\text{functions } \varphi : C_0 \to \mathbb{Z}\}$$

determined by $\varphi(a), \varphi(b), \varphi(c)$

$$C^1 = \{\text{functions } \psi : C_1 \to \mathbb{Z}\}$$
$$C^2 = \{\text{functions } \gamma : C_2 \to \mathbb{Z}\}$$

C^0

$$\delta\varphi(x) = \varphi(\partial x) = \varphi(b-a) = \varphi(b) - \varphi(a)$$

C_1

Started from a function φ on vertices, get a

Started from a function φ on vertices, get a new function $\delta\varphi$ on edges.

$$\delta\varphi(y) = \varphi(c) - \varphi(b)$$

$$\delta\varphi(z) = \varphi(a) - \varphi(c)$$

C'

$$\delta\psi(\Delta) = \psi(x) + \psi(y) + \psi(z)$$

Cocycles: $\delta\varphi = 0$ (φ = cocycle)

if $\varphi(a) = \varphi(b) = \varphi(c)$

Cohomology $H^0 = \text{Span}\langle \varphi : \varphi(a) = \varphi(b) = \varphi(c) \rangle$

$\delta\psi = 0$ if $\psi(x) + \psi(y) + \psi(z) = 0$

In this case ψ = coboundary $= \delta\varphi$ for some φ

How to find this φ?

Choose $\varphi(a) = 0$

want $\varphi(b) - \varphi(a) = \psi(x)$

$\Rightarrow \varphi(b) = \psi(x)$

want $\varphi(c) - \varphi(b) = \psi(y)$

$\Rightarrow \varphi(c) = \psi(x) + \psi(y)$

Check: $\varphi(a) - \varphi(c) = -(\psi(x) + \psi(y))$

Check: $\varphi(a) - \varphi(c) = -(\psi(x) + \psi(y))$

$= (\text{since } \delta\psi = 0) = \psi(z)$

$\Rightarrow$ conclude that $\delta\varphi = \psi$.

$\Rightarrow$ $H^1 = 0$.

It is easy to see that $H^2 = 0$.

<u>Conclusion</u>:

$$\left(\begin{array}{ll} H_0 = \mathbb{Z} & H^0 = \mathbb{Z} \\ H_1 = 0 & H^1 = 0 \\ H_2 = 0 & H^2 = 0 \end{array}\right\} \text{just computed}$$

contractible

$\Rightarrow H_* = H_*(pt)$

<u>Ex</u>

$$C_1 = \mathbb{Z} \xrightarrow{\cdot n} C_0 = \mathbb{Z} \qquad \text{chain cpx}$$

$x \qquad a$

$\partial(x) = n \cdot a$

$$C^0 \xrightarrow{\delta} C^1$$

$\varphi \qquad \psi$

$\delta\varphi(x) = \varphi(\partial x) = \varphi(na) = n\varphi(a)$

$\delta\varphi = 0$ if $\varphi(a) = 0 \Rightarrow H^0 = 0$

$$H^1 = C^1/\operatorname{Im}\delta = \frac{\langle \text{all } \psi \rangle}{\langle \text{all } \psi \text{ divisible by } n \rangle} = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$$

$H_0 = \mathbb{Z}_n \qquad H^0 = 0$

$H_1 = 0 \qquad H^1 = \mathbb{Z}_n$

$$H_1 = 0 \qquad H^1 = \mathbb{Z}_n$$

But if we work over a field then H_* and H^* have same dimension in this example.

<u>Case 1</u>: $n \neq 0$ in $\mathbb{K}$ (char $\mathbb{K} \nmid n$)

then $H_0 = H_1 = H^0 = H^1 = 0$ over $\mathbb{K}$.

$$\mathbb{K} \xrightarrow{\cdot n} \mathbb{K} \quad \text{acyclic}$$

<u>Case 2</u>: $n = 0$ in $\mathbb{K}$ (char $\mathbb{K} \mid n$)

$$C_\bullet = \mathbb{K} \xrightarrow{0} \mathbb{K} \qquad C^\bullet = \mathbb{K} \xrightarrow{0} \mathbb{K}$$

$$H_0 = H_1 = H^0 = H^1 = \mathbb{K}$$

<u>Conclusion</u>: 1) In this example, H_* and H^* depend on the choice of coefficients.

2) If we work over $\mathbb{Z}$, H_* and H^* are different in a tricky way (more details next time).

<u>Ex</u> $C_1 \xrightarrow{\partial} C_0$ two-term chain complex over $\mathbb{K}$

$=$ 1

complex over $\mathbb{K}$

$$H_0 = C_0/\mathrm{Im}\,\partial \qquad H_1 = \mathrm{Ker}\,\partial$$

$$C^0 \xrightarrow{\delta = \partial^*} C^1$$

$$H^0 = \mathrm{Ker}(\partial^*) \qquad H^1 = C^1/\mathrm{Im}\,\partial^*$$

<u>But:</u> $\mathrm{Ann}(\mathrm{Im}\,\partial) = \mathrm{Ker}(\partial^*)$ from HW problem

$$\dim \mathrm{Ker}(\partial^*) = \dim C_0 - \dim(\mathrm{Im}\,\partial)$$
$$= \dim C_0/\mathrm{Im}\,\partial$$

$$\Rightarrow \dim H_0 = \dim H^0$$

$$\mathrm{Ann}(\mathrm{Ker}\,\partial) = \mathrm{Im}(\partial^*)$$

$$\dim C^1/\mathrm{Im}\,\partial^* = \dim C^1 - \dim(\mathrm{Im}\,\partial^*)$$
$$= \dim(\mathrm{Ker}\,\partial).$$

$$\Rightarrow \dim H^1 = \dim H_1.$$
