# Lecture 10 (4/19)

Monday, April 19, 2021 1:49 PM

$X = n$-dim. manifold (Connected)!

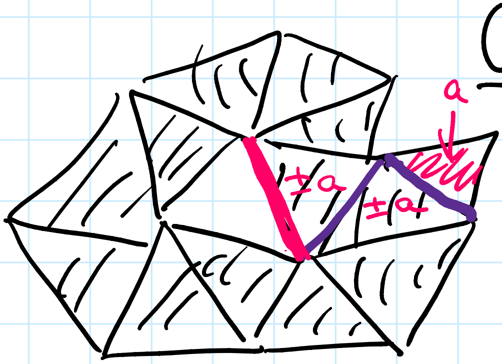
**Fact** $H_n(\mathring{X}) = 0$

Assume for now that $X$ is a PL manifold, that is, has a triangulation. Then $\mathring{X} = X -$ one top-dim simplex

(if $X$ is a surface, remove one triangle)

$$0 \longrightarrow C_n(\mathring{X}) \xrightarrow{\partial} C_{n-1}(\mathring{X}) \to \ldots$$

$$H_n(\mathring{X}) = \operatorname{Ker} \partial$$




Claim: If $X$ is connected, any nonzero cycle in $\operatorname{Ker}\partial$ must have all triangles in $\mathring{X}$ with nonzero coefficients. If one triangle  has nonzero coefficient $a$, then all other triangles have coefficients $\pm a \neq 0$. In particular, all simplices adjacent to the one we removed have nonzero coefs

$\Rightarrow$ if we apply $\partial$, $(n-1)$-dim faces of the missing simplex appear with nonzero coefs

$$\Rightarrow \partial\left(\sum (\pm a)\,\Delta_i\right) \neq 0$$

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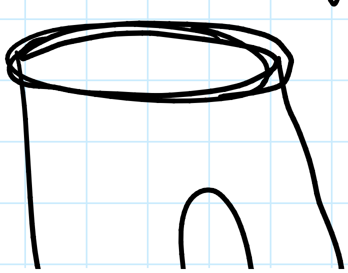
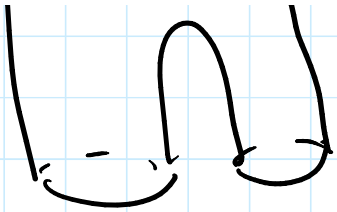
## Proof of classification of surfaces

Pair of pants $= S^2$ with 3 holes



= orientable surface connecting one circle

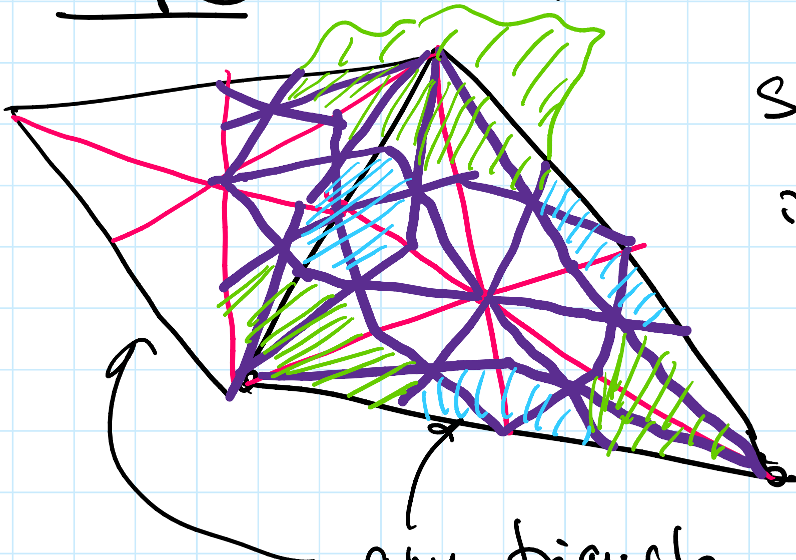


connecting one circle
on top with 2 circles
on the bottom

**Thm** Any orientable surface (PL) connected compact
is homeomorphic to $\underbrace{T^2 \# T^2 \# \ldots \# T^2}_{g}$

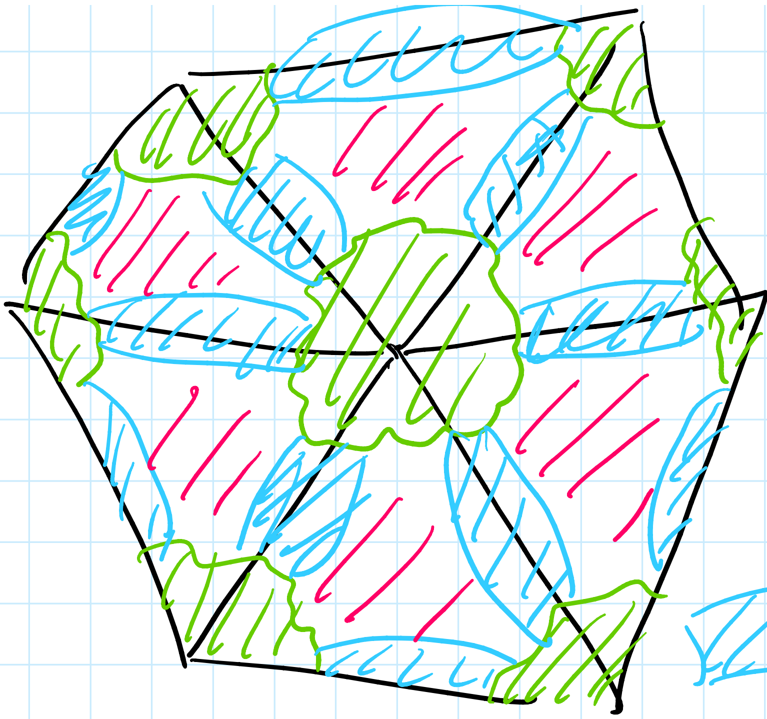
**Proof** $\Sigma$ = our surface, we choose
a triangulation

Step ①: Double baricentric subdivision

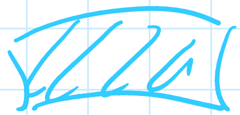


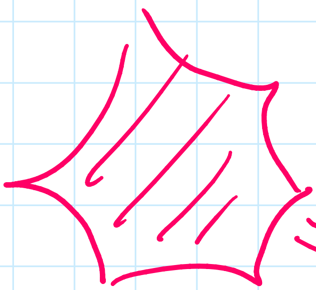

Split each triangle into 6 smaller triangles and repeat for each of them.

We can combine the triangles in double baricentric subdivision as follows:



 = disks near each vertex of the original triangulation

 = ribbon near each edge of the original triangulation

 = disks in the middle of each face of the original triangulation

Step 2 From this, we reconstruct our surface as follows:

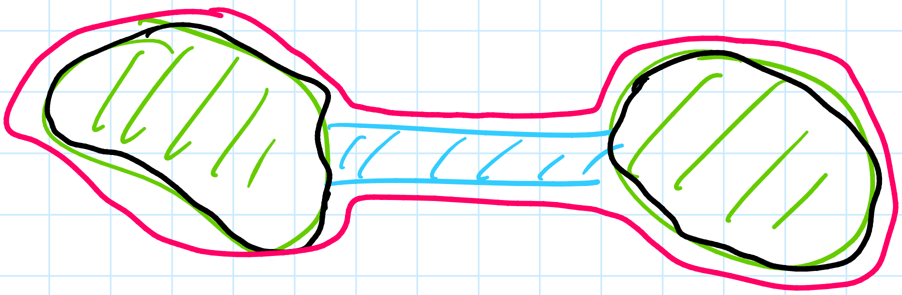
2a Draw green disks

$\sqcup D^2_i$

2b Add blue ribbons connecting them (will understand better soon) them (will understand better soon)

(2c) Add the red disks.

(Step 3) Understand better what happens when we add a ribbon in 2b.



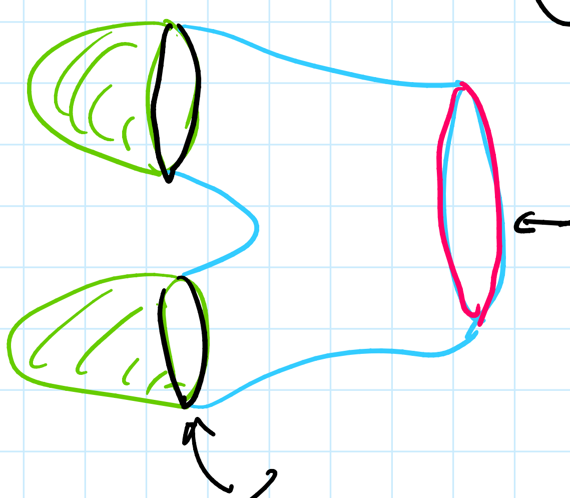
black = bdry.

red = new bdry

Case 1: New ribbon connects two disjoint components of a surface.

Before: 2 boundary components,

after: 1 boundary componen

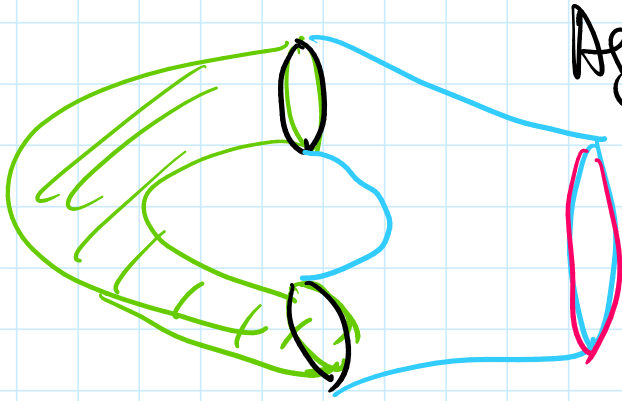



Can visualize pair of pants by lifting red boundary above the surface.

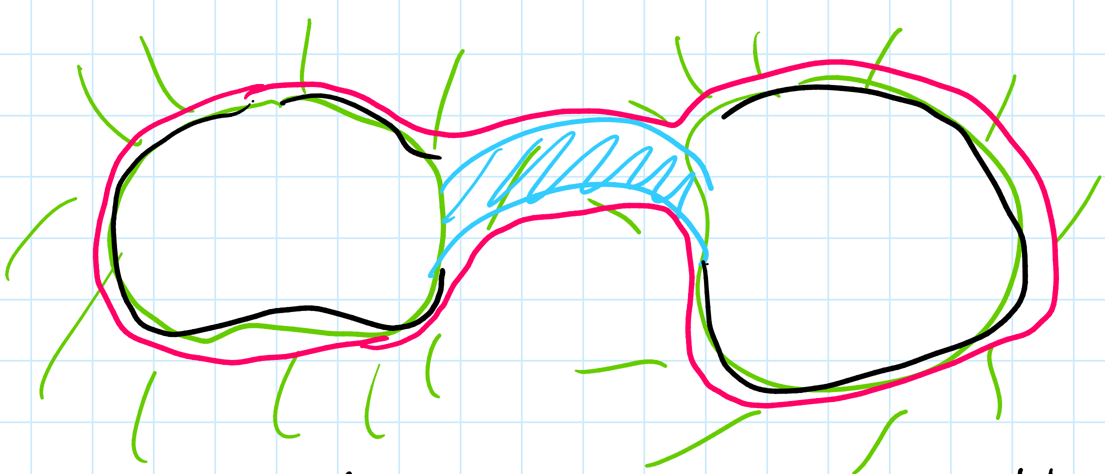
This is a pair of pants, connecting

This is a pair of pants, connecting 2 components into one.

Case 2: New ribbon connects two different boundary components into one. (Could be in the same connected component of a surface)

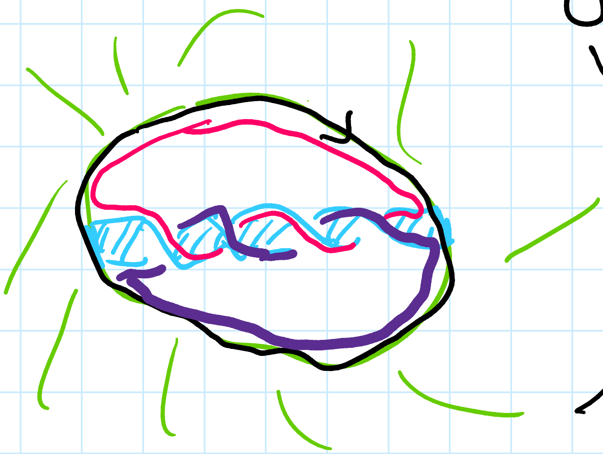


Again we add a pair of pants which connects two boundary components with



Case 3 The ends of the ribbon are on the same connected component

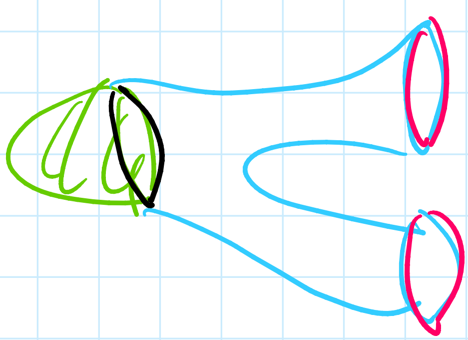
on the same connected component of the boundary of the surface that



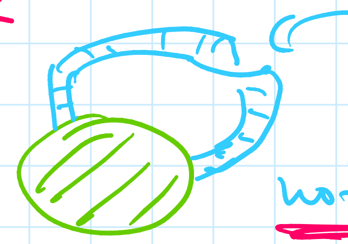
we constructed before this step. This surface is <u>oriented</u>

The orientation on the ribbon should agree with orientation on the boundary $\Rightarrow$ the ribbon has <u>even</u> number of twists.

Therefore, after gluing in the ribbon, we replaced one boundary component with two $\Rightarrow$ new piece of surface is homeomorphic to a pair of pants.



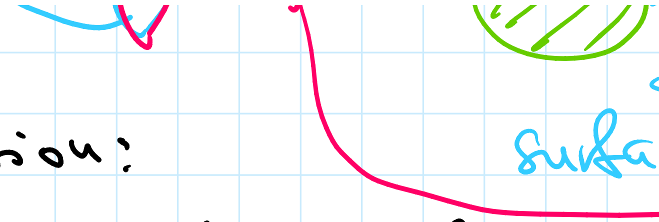
Note



Mobius band

<u>not allowed</u>

since our

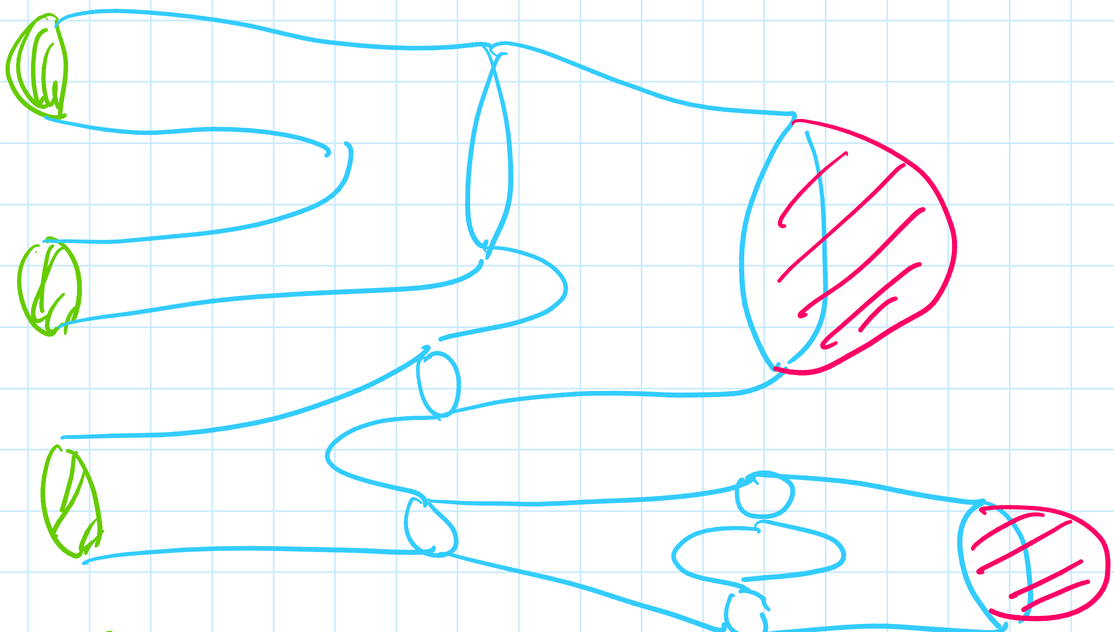


not allowed

since our surface is oriented.

Step 4 Conclusion:

we decomposed the surface into cups, caps and pairs of pants:



green disks in the beginning = cups.

blue ribbons $\longleftrightarrow$ pairs of pants (assuming $\Sigma$ oriented)

red disks $\longleftrightarrow$ caps.

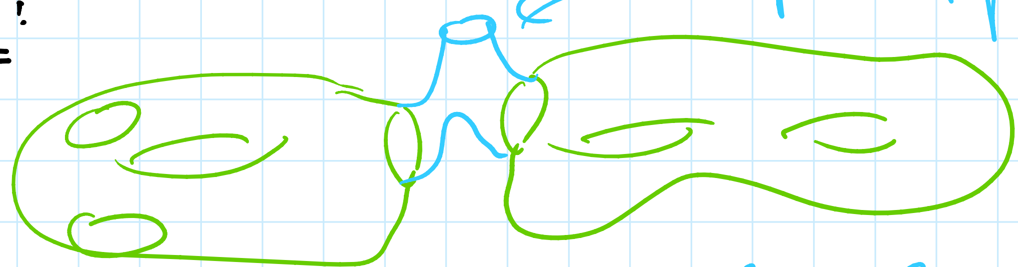
Theorem follows if we prove that the result is a genus $g$ surface (= connected sum $\underbrace{T^2 \# T^2 \ldots \# T^2}_{g}$)

$\underbrace{\qquad\qquad\qquad\qquad}_{g}$

We can prove that on each step we have (possibly disconnected) genus $g$ surface with some number of holes.

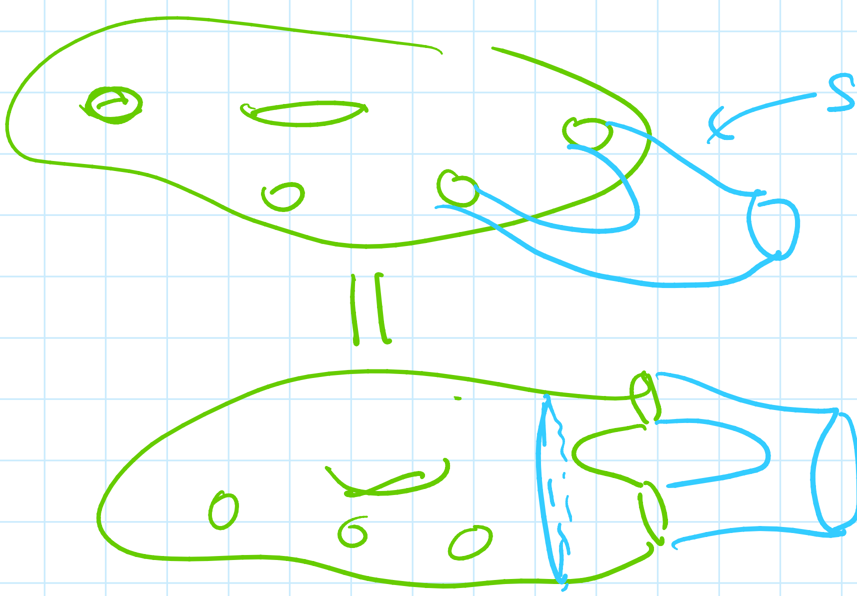
<u>Base</u>: green disks ($= S^2$ with one hole)

<u>Step</u>:

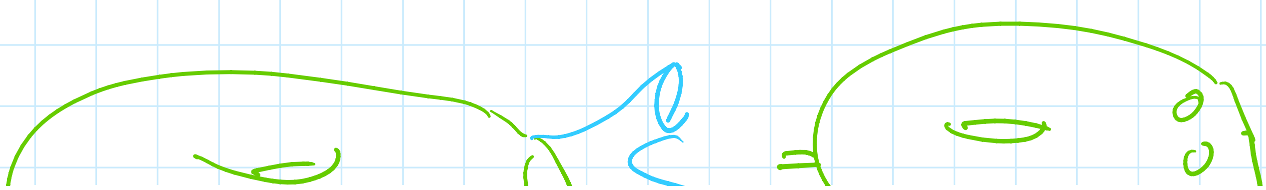


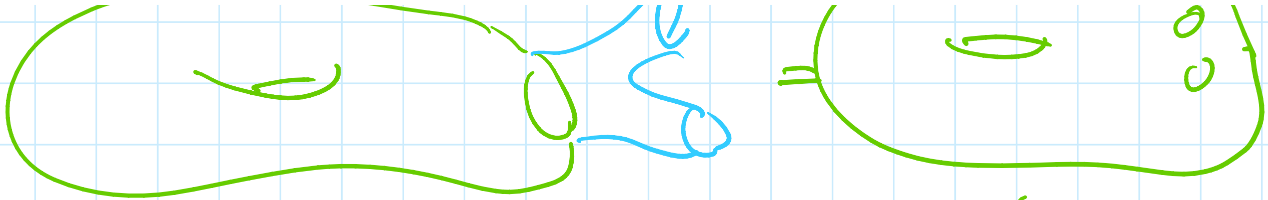
← add pair of pants.

(⟹) connect sum of surfaces + make a hole.



← same as connect sum with $T^2$ and add a hole.





create new hole

In the end, we fill in the holes with red disks.

=> get a closed surface with no holes.