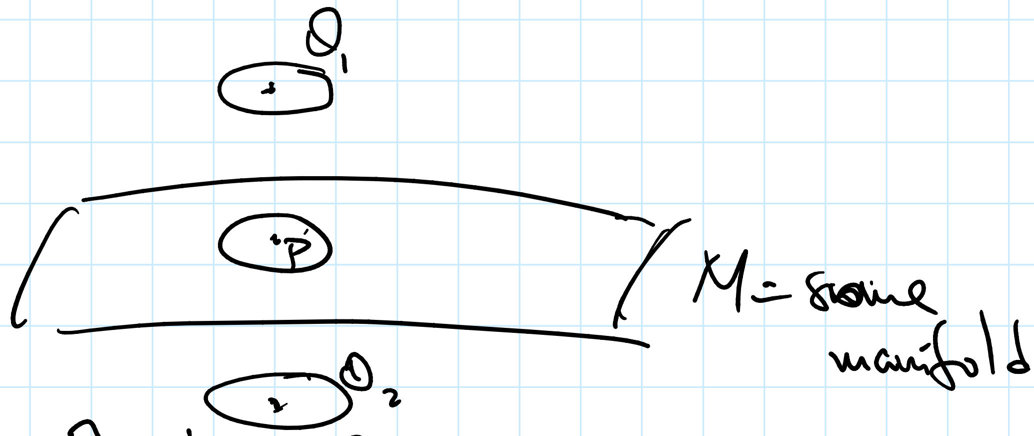
Lecture 12 (4/23)

Friday, April 23, 2021 2:07 PM

Orientation cover (from HW)



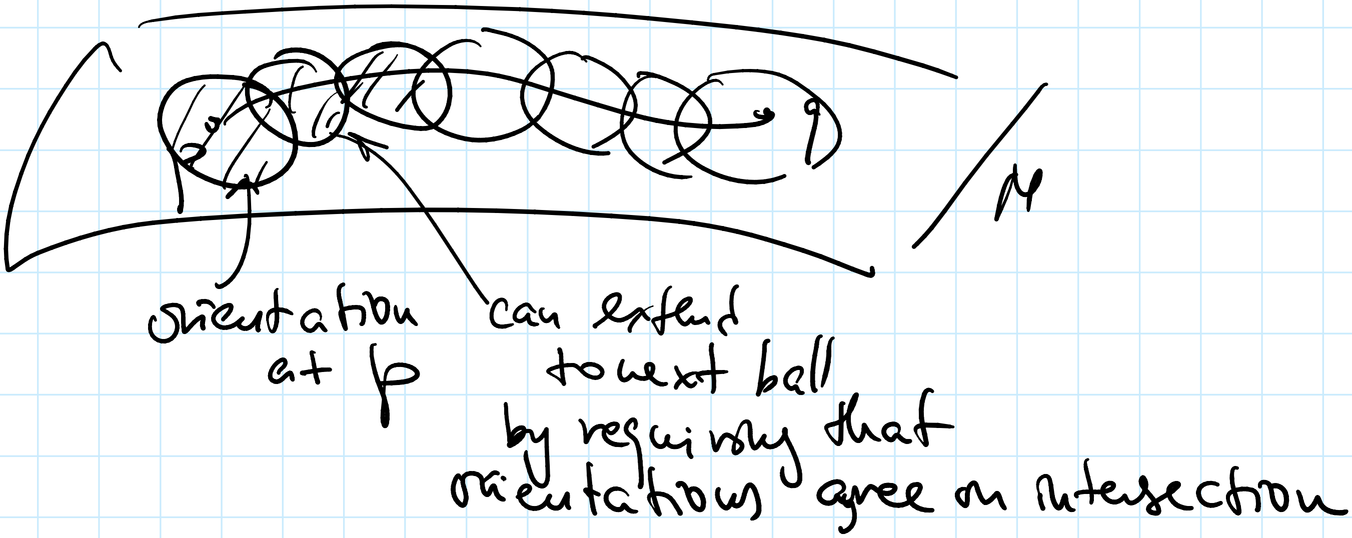
$\tilde{M} = \{(p, \theta)$; p = point in M
θ = local orientation at p $\}$

<u>Key idea #1</u>: for a small neighborhood of p (a ball in $\mathbb{R}^n$) there are exactly two orientations which agree for all points in this nbhd.

Small nbhd in $M \longleftrightarrow$ two small nbhds in $\tilde{M}$ which are homeomorphic to balls.

<u>Key idea # 2</u>: can transport

orientation along a path:



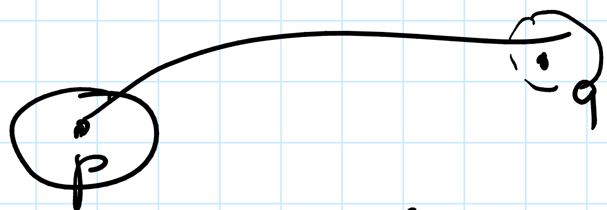
Idea: can extend orientation uniquely along a path.

Lemma M is orientable if connected and only if

(a) Transporting orientation from p to q along a path does not depend on a path.

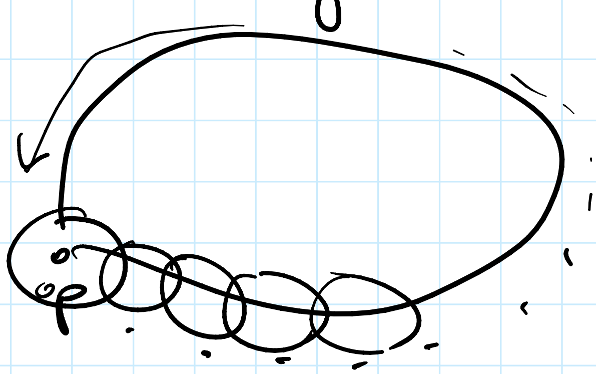
Proof: $\Rightarrow$ if M is orientable, define orientation at every point consistently this agrees with transport along any path

$\Leftarrow$ If transport does not depend on a path, <u>choose</u> some orientation at p



Define orientation at q using any path from p to q. $\square$

(b) M <u>connected</u>. M is orientable if and only if for any loop in M transport of the orientation agrees with the original one.



<u>Fact</u> (b) $\Longleftrightarrow$ (a).

Two kinds of loops in M:

two kinds of loops in M:

- orientation preserving
- orientation reversing (as in Möbius band)

(b): M is orientable iff no orientation reversing loops.

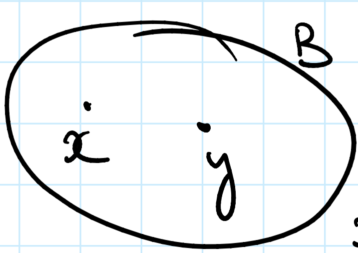
HW hint: how do these loops lift to the double cover $\tilde{M}$?

M = topological n-manifold

a local orientation at $x \in M$ is a generator of local homology group $H_n(M; M-\{x\}; \mathbb{Z}) \ni \alpha_x$

IS last time

$\tilde{H}_{n-1}(S^{n-1}, \mathbb{Z}) \simeq \mathbb{Z}$



x, y in same ball B

local orientations at x and y agree if there's a class in $H_n(M; M-B)$

there's a class α in $H_n(M; M-B)$

$$\begin{tikzcd} & H_n(M; M-B) \arrow[dl] \arrow[dr] & \\ \alpha_x \in H_n(M; M-\{x\}) & & H_n(M; M-\{y\}) \ni \alpha_y \end{tikzcd}$$

$\Rightarrow$ α_x agrees with α_y.

Thm (a) If M orientable then

$H_n(M; \mathbb{Z}) \simeq \mathbb{Z}$

M compact, connected

(b) If M is not orientable then

$H_n(M; \mathbb{Z}) \simeq 0$

(c) If M is any n-manifold

$H_n(M; \mathbb{Z}_2) = \mathbb{Z}_2$.

Proof (b) Suppose $H_n(M; \mathbb{Z}) \neq 0$

$$\begin{array}{c} \Downarrow \\ \alpha \end{array}$$

$$H_n(M; \mathbb{Z}) \longrightarrow H_n(M; M-\{x\})$$

$$\alpha \longrightarrow \alpha_x$$

This defines a class α_x for all x

clearly agrees for all points x

(note: if $\alpha \neq 0$ then $\alpha_x \neq 0$ from exact sequence of a pair).

$\Rightarrow$ M is orientable.

(a) Conversely, assume that M is orientable. We want to prove that $H_n(M) \simeq \mathbb{Z}$, prove a more general statement:

(*) $H_n(M; M - A) \simeq \mathbb{Z}$

for some compact subsets $A \subset M$

If we prove this, we are done:

$$H_n(M) \simeq H_n(M; \emptyset) = H_n(M; M - M)$$

Can prove (*) "by induction":

- If A is a closed ball in some chart ($\simeq \mathbb{R}^n$)

then $H_n(M; M - A) \overset{\text{excision}}{=} H_n(\mathbb{R}^n, \mathbb{R}^n - A)$

$\simeq \mathbb{Z}$

same as last time.

- If $(*)$ for A and B, this is true for $A \cup B$.

This follows from Mayer-Vietoris sequence: $M - (A \cup B) = (M - A) \cap (M - B)$

$$0 \to H_n(M; M - (A \cup B)) \to \underbrace{H_n(M; M-A) \oplus H_n(M; M-B)}_{\mathbb{Z}\oplus\mathbb{Z} \text{ by assumption}} \to \underbrace{H_n(M; M - (A \cap B))}_{\mathbb{Z}} \to \cdots$$

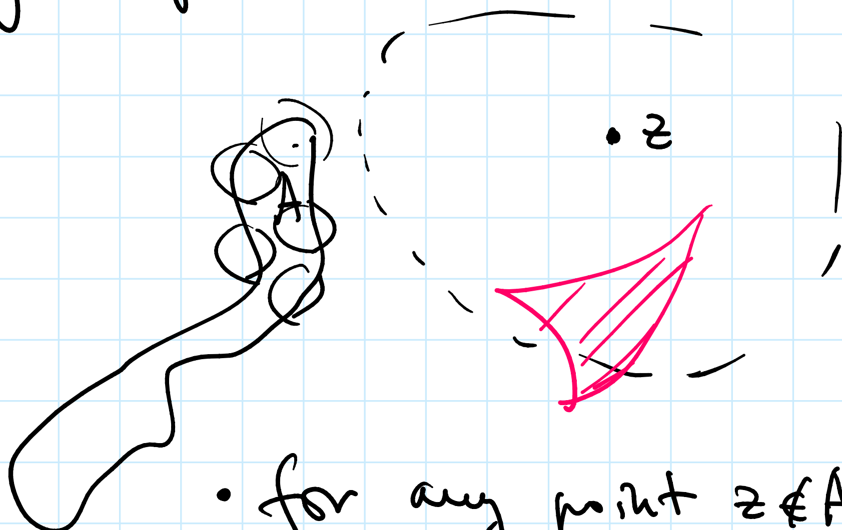
$\Rightarrow$ exact sequence

$$0 \to ? \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(1,\,1)} \mathbb{Z} \to \cdots$$

$\Rightarrow ? \simeq \mathbb{Z}$.

. A = ball ~~or~~ (or convex subset of $\mathbb{R}^n$)

- Any union of closed balls in $\mathbb{R}^n$ (or convex subsets), is fine.
- Any compact subset A in $\mathbb{R}^n$



- for any point $z \notin A$

$a \in A$ there is a neighborhood of a not covering z

These give an open cover of A, since A is compact, we can choose a finite subcover $\Rightarrow$ finite union of balls covers A does not cover z.

Any singular chain in $M - A$ has the same property (finite has the same property (finite union of simplices)

$\rightsquigarrow$ can replace A by a union of such balls without changing $H_n(M; M-A) = H_n(\mathbb{R}^n; \mathbb{R}^n - A)$.

- Note: intersection of balls is convex, contractible $\Rightarrow$

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \underset{\text{compact}}{(\text{convex})}) = H_n(\mathbb{R}^n - \text{ball}).$$

- Since M is compact, can cover it by finitely many balls $\Rightarrow$ true for M.

Conclusion: construct fundamental class in $H_n(M; \mathbb{Z})$ locally for balls, use Mayer-Vietoris to glue them to a class on the whole M.

(c) Proof is the same (over $\mathbb{Z}_2$)

M is R-orientable if we ~~a~~ can choose a generator in $H_n(M; M-\{x\}; \underline{R})$ consistently

<u>Thm</u> M is R-orientable $\Longleftrightarrow$

$$H_n(M; R) = R$$

so $\mathbb{Z}$-orientable $\Leftrightarrow$ orientable

any manifold is $\mathbb{Z}_2$-orientable.

<u>Next week</u>: Poincaré duality!

If M is R-orientable then

$$\boxed{H_k(M; R) \simeq H^{n-k}(M; R)}$$

If R is a field

$$\dim H_k(M; R) \simeq \dim H_{n-k}(M; R)$$

$$= \dim H^{n-k}(M; R).$$

$k=0$ $\dim H_0 = \dim H_n = 1$ if M is connected

$k=0$ $\dim H_0 = \dim H_n = 1$ if M is connected and R-orientable.