Thm M = orientable mfd of dim $= n$

$$H_n(M;\mathbb{Z}) \simeq \mathbb{Z}$$

$[M]$ = fundamental class = generator

[depends on choice of orientation]

Applications:

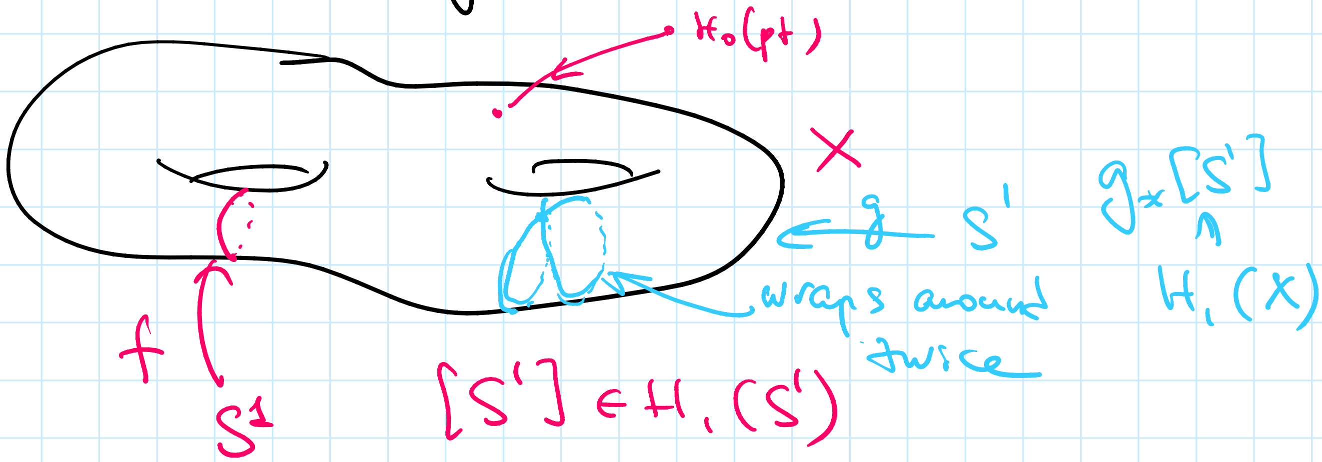
(1) $M \xrightarrow{f} X$

M: oriented n-mfd; X ← some top. space

$[M] \in H_n(M)$ $\qquad f_*: H_n(M) \to H_n(X)$

$f_*[M] \in H_n(X)$

interesting class in $H_n(X)$!



$[S^1] \in H_1(S^1)$

$f_*[S^1] \in H_1(X)$ = class of this circle.

Realization problem: which classes in $H_*(X)$ can be obtained this way?

Answer (Thom) • Some classes cannot be realized by smooth manifolds

• $\alpha \in H_n(X)$ then for some $m \in \mathbb{Z}$, $m \neq 0$, $m\alpha$ can be realized by a manifold.
n-any

Hard, cobordism theory.

Easy exercise: • Any class in H_1(surface) can be realized by a map $S^1 \xrightarrow{f} X$

• Any class in $H_1(X)$ can be realized by a map $S^1 \longrightarrow X$

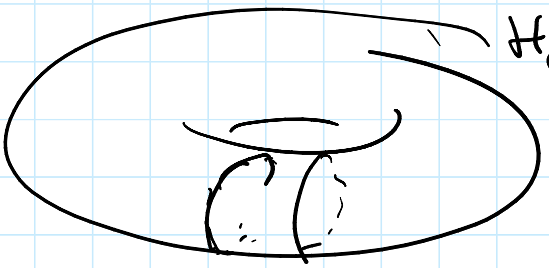
$$H_1(X) = \frac{\pi_1(X)}{[\pi_1, \pi_1]}$$

Any element in $\pi_1(X)$ = loop in X

$\doteq$ map $\gamma: S^1 \to X$

$\gamma_* [S^1]$ = corresponding class in $H_1(X)$

$H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$



$H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$

(a, b)

$\gcd(a,b) = 1 \Rightarrow$ can draw an embedded curve in T^2

$\gcd(a,b) = d > 1 \Rightarrow d$ curves with slope $\left(\frac{a}{d}, \frac{b}{d}\right)$

Special case: <u>submanifolds</u>

$N^k \subset M^n$

chart in M

chart in N

N locally looks like a coordinate subspace $\mathbb{R}^k \subset \mathbb{R}^n$

$\varphi : U \longrightarrow \mathbb{R}^n$ chart.

$U \cap N = \varphi^{-1}(\mathbb{R}^k)$

Clearly, if M is smooth and N is a submanifold then N is a smooth k-manifold.

If N is orientable then

If N is orientable then

$$[N] := i_*[N] \in H_k(M)$$

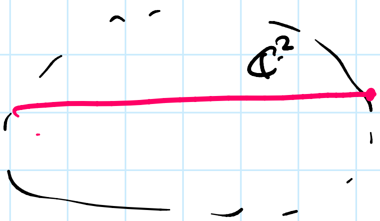
$$i : N \hookrightarrow M$$

inclusion.

fund. class of a submanifold.

Ex $\mathbb{C}P^1 \subset \mathbb{C}P^2$

$\mathbb{C}P^2 = \mathbb{C}^2 \cup$ infinite line



$$i_*[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$$

$\parallel$

generator of $H_2(\mathbb{C}P^2)$

since $\mathbb{C}P^1 = (2\text{-cell}) \cup (0\text{-cell})$ in $\mathbb{C}P^2$

$$[\mathbb{C}P^1] = [2\text{-cell}] \in H_2(\mathbb{C}P^2), \ H_2(\mathbb{C}P^2).$$

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(2) Poincaré duality, first approach

M = orientable n-manifold

$$\alpha \in H^i(M) \qquad \beta \in H^{n-i}(M)$$

$$\alpha \cup \beta \in H^n(M)$$

$$(\alpha, \beta) = \alpha \cup \beta([M]) \in \mathbb{Z}$$

evaluate $\alpha \cup \beta$ on fund. class

evaluate $\alpha \cup \beta$
on fund. class

We get a bilinear pairing

$$(\alpha,\beta): H^i(M) \times H^{n-i}(M) \longrightarrow \mathbb{Z}$$

Same works over $\mathbb{Z}_2$, does not need to be orientable.

Thm (Poincaré duality) M = connected, oriented n-mfd

$\mathbb{k}$ = any field

$$(\alpha,\beta) = \alpha \cup \beta\,([M]) : H^i(M) \times H^{n-i}(M) \to \mathbb{k}$$

is nondegenerate: vector spaces over $\mathbb{k}$

- if $(\alpha,\beta) = 0$ for all $\beta \Rightarrow \alpha = 0$
- if $(\alpha,\beta) = 0$ for all $\alpha \Rightarrow \beta = 0$

Lemma If U, V = vector spaces and $(\cdot,\cdot): U \times V \longrightarrow \mathbb{k}$ is a nondegenerate bilinear pairing then $V \simeq U^*$.

Proof Fix $\alpha \in U$, define a function $\phi_\alpha : V \longrightarrow \mathbb{k}$ $\quad \phi_\alpha \in V^*$

$$\phi_\alpha(v) = (\alpha, v).$$

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This defines a map $\phi : U \longrightarrow V^*$
$\alpha \longrightarrow \phi_\alpha$

$\phi_\alpha = 0 \Rightarrow (\alpha, v) = 0$ for all $v \in V$

Since $(\cdot,\cdot)$ is nondegenerate, $\alpha = 0$

$\Rightarrow \operatorname{Ker}\phi = 0$, ϕ is injective and $\dim U \leq \dim V^*$

Similarly, $\dim V \leq \dim U^*$

$$\begin{matrix} \dim V & \leq & \dim U^* \\ \| & & \| \\ \dim V^* & \geq & \dim U \end{matrix}$$

$\Rightarrow \dim U = \dim V \Rightarrow \phi$ is an isomorphism.

Cor: • $H^i(M; \mathbb{K}) \simeq \left(H^{n-i}(M; \mathbb{K})\right)^*$!

$$\left(H^{n-i}(M;\mathbb{K})\right)^* = H_{n-i}(M, \mathbb{K})$$

$$\boxed{H^i(M; \mathbb{K}) \simeq H_{n-i}(M; \mathbb{K})}$$

• $\dim H^i = \dim H^{n-i} = \dim H_{n-i}$

Dimensions of H^i / H_i are symmetric around the middle.

• Example $\dim H^0 = \dim H^n = 1$

• If $n = 2k$ is even then we have

$$H^k(M) \times H^k(M) \longrightarrow \mathbb{Z}$$

intersection form in the middle cohomol

symmetric if k is even

skew-symmetric if k is odd

$$\beta \cup \alpha = (-1)^{k \cdot k} \alpha \cup \beta \qquad \alpha, \beta \in H^k$$

M = surface $n=2, k=1$

$$\Rightarrow H^1(M) \times H^1(M) \longrightarrow \mathbb{Z}$$

$$(\alpha, \beta) = -(\beta, \alpha)$$

M = 4-manifold $n=4, k=2$

$$H^2(M) \times H^2(M) \longrightarrow \mathbb{Z}$$

$$(\alpha, \beta) = (\beta, \alpha)$$

symmetric nondegenerate bilinear form.

<u>Ex</u>: $M = T^2$ $\quad \langle \underbrace{1}_{H^0}, \underbrace{\alpha, \beta}_{H^1}, \underbrace{\alpha \cup \beta}_{H^2} \rangle$ $\quad \alpha \cup \beta = -\beta \cup \alpha$

$$H^0 \times H^2 \longrightarrow \mathbb{Z}$$

$$(1, \alpha \cup \beta) \longrightarrow \alpha \cup \beta([T^2]) = 1$$

can choose orientation such that it holds.

H^0 is dual to H^2

$\alpha \cup \beta$ = generator of H^2

$$H^1 \times H^1 \longrightarrow \mathbb{Z}$$

$\alpha \cup \beta$ = generator of H^2

eval on $[T^2] = 1$

$$H^1 \times H^1 \longrightarrow \mathbb{Z}$$

$$(\alpha, \beta) = \alpha \cup \beta([T^2]) = 1$$

$$(\alpha, \alpha) = \alpha \cup \alpha([T^2]) = 0$$

$$(\beta, \beta) = \beta \cup \beta([T^2]) = 0$$

$$(\beta, \alpha) = \beta \cup \alpha([T^2]) = -\alpha \cup \beta([T^2]) = -1$$

$$\begin{array}{c c} & \begin{array}{cc} \alpha & \beta \end{array} \\ \begin{array}{c} \alpha \\ \beta \end{array} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array} = \text{matrix of the intersection form on } H^1(T^2).$$

Ex: T^4

dual bases (up to signs).

H^0	H^1	H^2	H^3	H^4
1	α_1	$\alpha_1 \cup \alpha_2$	$\alpha_1 \cup \alpha_2 \cup \alpha_3$	$\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$
	α_2	$\alpha_1 \cup \alpha_3$	$\alpha_1 \cup \alpha_2 \cup \alpha_4$	
	α_3	$\alpha_1 \cup \alpha_4$	$\alpha_1 \cup \alpha_3 \cup \alpha_4$	
	α_4	$\alpha_2 \cup \alpha_3$	$\alpha_2 \cup \alpha_3 \cup \alpha_4$	
		$\alpha_2 \cup \alpha_4$		
		$\alpha_3 \cup \alpha_4$		

$$\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4([T^4]) = 1$$

$$H^1 \times H^3 \longrightarrow \mathbb{Z}$$

$$\alpha_1 \cup (\alpha_1 \cup \alpha_2 \cup \alpha_3)([T^2]) = 0$$

$$\alpha_1 \cup (\alpha_1 \cup \alpha_2 \cup \alpha_4)([T^2]) = 0$$

$$\alpha_1 \cup (\alpha_1 \cup \alpha_3 \cup \alpha_4)([T^2]) = 0$$

$$\alpha_1 \cup (\alpha_2 \cup \alpha_3 \cup \alpha_4)([T^2]) = 1.$$

$\alpha_1 \cup (\alpha_2 \cup \alpha_3 \cup \alpha_4)([T^2]) = 1.$

$$\alpha_2 \cup (\alpha_1 \cup \alpha_3 \cup \alpha_4)([T^2]) =$$

$$= -\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4([T^2]) = -1.$$

$H^2 \times H^2 \to \mathbb{Z}$ next time

$$(\alpha_1 \cup \alpha_2, \alpha_3 \cup \alpha_4) = (\alpha_3 \cup \alpha_4, \alpha_1 \cup \alpha_2) = 1$$

by sign rule $(-1)^{2\cdot 2} = 1$.

Künneth: $X = n$-mfd, $Y = l$-mfd. $\}$ orientable

$$[X \times Y] \in H_{n+l}(X \times Y)$$
$$\| \quad \text{by Künneth.}$$
$$[X] \otimes [Y]$$
$$[X] \in H_n(X), \quad [Y] \in H_l(Y)$$

More next time.