Lecture 14 (4/28)

Wednesday, April 28, 2021 2:09 PM

Some remarks:

(1) Künneth formula and fundamental class

X = k-manifold Y = n-manifold

connected, orientable

$X \times Y$ = $k+n$-manifold

$\underset{\mathbb{R}^{k+n}}{\text{charts}} = \underset{\mathbb{R}^k}{(\text{chart in } X)} \times \underset{\mathbb{R}^n}{(\text{chart in } Y)}$

$X \times Y$ orientable

$[X] \in \underset{\mathbb{Z}}{H_k(X)}$ $[Y] \in \underset{\mathbb{Z}}{H_n(Y)}$ fund. classes

$[X] \otimes [Y] \in \underset{\mathbb{Z}}{H_k(X)} \otimes \underset{\mathbb{Z}}{H_n(Y)} \underset{\text{Künneth}}{\simeq} \underset{\mathbb{Z}}{H_{k+n}(X \times Y)}$

$[X] \otimes [Y]$ is a generator of

$H_{k+n}(X \times Y) \Rightarrow [X] \otimes [Y] = [X \times Y]$

Warning: If we swap X and Y,

Warning: If we swap X and Y, we change orientation on $X \times Y$ by $(-1)^{kn}$

② Intersection form on H^* and Künneth

α, β = classes in $H^*(X)$

γ, δ = classes in $H^*(Y)$

$$(\alpha,\beta) = \alpha \cup \beta\,([X])$$

$$(\gamma,\delta) = \gamma \cup \delta\,([Y])$$

$$\boxed{(\alpha\otimes\gamma, \beta\otimes\delta)} = (\alpha\otimes 1)\cup(1\otimes\gamma)\cup(\beta\otimes 1)\cup(1\otimes\delta)[X\times Y]$$

$$= (-1)^{\deg\beta\deg\gamma}(\alpha\otimes 1)\cup(\beta\otimes 1)\cup(1\otimes\gamma)\cup(1\otimes\delta)[X\times Y]$$

$$= (-1)^{\deg\beta\deg\gamma}(\alpha\cup\beta\otimes 1)\cup(1\otimes\gamma\cup\delta)\,[X\times Y]$$

$$= (-1)^{\deg\beta\deg\gamma}\,(\alpha\cup\beta)\otimes(\gamma\cup\delta)\big([X]\otimes[Y]\big) =$$

$$= (-1)^{\deg\beta\deg\gamma}\,(\alpha\cup\beta)[X]\cdot(\gamma\cup\delta)[Y] =$$

$$\boxed{= (-1)^{\deg\beta\deg\gamma}\,(\alpha,\beta)\cdot(\gamma,\delta).}$$

Ex $T^2 = S^1 \times S^1$

$\underset{H^*}{1,\alpha}$ $1,\beta$

$$(\alpha \otimes 1, 1 \otimes \beta) =$$

$$= \alpha \otimes \beta \left([S^1] \otimes [S^1]\right)$$

$$= \alpha([S^1]) \cdot \beta([S^1]) = 1 \cdot 1 = 1$$

<u>Ex:</u> $T^4 = \underset{\alpha_1}{S^1} \times \underset{\alpha_2}{S^1} \times \underset{\alpha_3}{S^1} \times \underset{\alpha_4}{S^1}$

H^0	H^1	H^2	H^3	H^4
1	α_1	$\alpha_1 \cup \alpha_2$	$\alpha_1 \cup \alpha_2 \cup \alpha_3$	$\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$
	α_2	$\alpha_1 \cup \alpha_3$	$\alpha_1 \cup \alpha_2 \cup \alpha_4$	
	α_3	$\alpha_2 \cup \alpha_4$	$\alpha_1 \cup \alpha_3 \cup \alpha_4$	
	α_4	$\alpha_2 \cup \alpha_3$	$\alpha_2 \cup \alpha_3 \cup \alpha_4$	
		$\alpha_2 \cup \alpha_4$		
		$\alpha_3 \cup \alpha_4$.		

$$\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4 (T^4) = 1 = \underset{1}{\alpha_1[S^1]} \cdot \underset{1}{\alpha_2[S^1]} \cdot \underset{1}{\alpha_3[S^1]} \cdot \underset{1}{\alpha_4[S^1]}$$

$(\alpha_1, (\alpha_2 \cup \alpha_3 \cup \alpha_4)) = 1$ since $\alpha_1 \cup (\alpha_2 \cup \alpha_3 \cup \alpha_4) =$

$H^1 \times H^3 \longrightarrow \mathbb{Z}$ $\qquad \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$

$$\alpha_2 \cup (\alpha_1 \cup \alpha_3 \cup \alpha_4) = -\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$$

and so on.

$\underbrace{H^2 \times H^2}_{\text{Matrix}} \longrightarrow \mathbb{Z}$

$$\begin{array}{c c} & \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \begin{array}{c} 12 \\ 13 \\ 14 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$(\alpha_1 \cup \alpha_2) \cup (\alpha_3 \cup \alpha_4)[T^4]$
$= 1$

$$\begin{array}{c|c} 13 & \\ 14 & \\ 23 & \\ 24 & \\ 34 & \end{array}\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$= 1$

$(\alpha_1 \cup \alpha_3) \cup (\alpha_2 \cup \alpha_4)$

$= -1$

$(\alpha_1 \cup \alpha_4) \cup (\alpha_2 \cup \alpha_3)$

$= +1$

Symmetric bilinear form,

$$(\alpha_3 \cup \alpha_4) \cup (\alpha_1 \cup \alpha_2) = (-1)^{2\cdot 2} \cdot (\alpha_1 \cup \alpha_2) \cup (\alpha_3 \cup \alpha_4) = +1$$

$n = 2k$ $\quad H^k(M) \times H^k(M) \longrightarrow \mathbb{Z}$ $\quad M$ = n-mfd

symmetric if k is even ($\Rightarrow 4 \mid n$)

antisymmetric if k is odd ($n = 4s+2$)

Rmk In 4-dimensional topology (and in higher dimensions) intersection form is a very important invariant. In particular, we can study its signature.

Rmk $\dim H^k(T^n) = \binom{n}{k}$ ← from HW

basis $\longleftrightarrow$ k-element subsets of $\{\alpha_1 \ldots \alpha_n\}$ S

of $\{\alpha_1, \dots, \alpha_n\}$ $\quad S$

$$= \dim H^{n-k}(T^n) = \binom{n}{n-k}$$

dual basis (wrt pairing) (up to sign) $\longleftrightarrow$ $\{1 \dots n\} - S$ complementary subset

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<u>Fact</u> (Poincaré duality over $\mathbb{Z}$)

The intersection form

$$H^k(M;\mathbb{Z}) \times H^{n-k}(M;\mathbb{Z}) \to \mathbb{Z}$$

$$(\alpha, \beta) = \alpha \cup \beta([M])$$

If α is a torsion class, $m\alpha = 0$

$$0 = (m\alpha, \beta) = m(\alpha, \beta) \Rightarrow (\alpha, \beta) = 0$$

$\Rightarrow (\alpha, \beta)$ kills the torsion completely

<u>Thm</u> On $H^k_{free} \times H^{n-k}_{free}$ this is a perfect pairing, that is,

$$H^k_{free} \xrightarrow{\sim} \left(H^{n-k}_{free}\right)^* = H_{n-k, free}$$

$$\alpha \longrightarrow \phi_\alpha$$

$$\phi_\alpha(\beta) = (\alpha, \beta)$$

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This is an isomorphism over $\mathbb{Z}$!

Ex $H^k = \mathbb{Z}\langle \alpha \rangle$ $H^{n-k} = \mathbb{Z}\langle \beta \rangle$

generator

then $(\alpha, \beta) = \pm 1$

$$\phi_\alpha(\beta) = (\alpha, \beta)$$

$$\phi_{m\alpha}(\beta) = m(\alpha, \beta)$$

$(\alpha, \beta) = 2$

$\phi_\alpha(\beta) = 2$

$\phi_\alpha(\beta) = 2$

$\phi_{k\alpha}(\beta) = 2k$

never get dual basis to β

$\Rightarrow$ the image of ϕ consists of elements in $(H^{n-k})^*$ divisible by (α, β)

Can get all elements of $(H^{n-k})^*$ iff $(\alpha, \beta) = \pm 1$.

Ex $H^k_{free} \times H^k_{free} \longrightarrow \mathbb{Z}$

$n = 2k$ $\boxed{\det = \pm 1}$ $H^k_{free} \simeq (H^k_{free})^*$

Thm $H^*(\mathbb{C}P^n) \simeq \mathbb{Z}[\alpha] / \alpha^{n+1} = 0$

$\alpha \in H^2$

Fact: $\mathbb{C}P^n$ is orientable! $\Leftarrow H_{2n}(\mathbb{C}P^n) = \mathbb{Z}$

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Proof: induction in n

$n=1$ $\quad \mathbb{C}P^1 \simeq S^2 \quad \mathbb{Z}[\alpha]/_{\alpha^2=0}$ ✓

Step: Suppose we know this for $\mathbb{C}P^{n-1}$

$$H^*(\mathbb{C}P^{n-1}) = \langle 1, \alpha, \alpha^2, \ldots \alpha^{n-1} \rangle$$

basis $\qquad \shortparallel \quad \dfrac{\mathbb{Z}[\alpha]}{(\alpha^n)}$

$$\mathbb{C}P^{n-1} \xhookrightarrow{i} \mathbb{C}P^n$$

$$i^* : H^*(\mathbb{C}P^n) \to H^*(\mathbb{C}P^{n-1})$$

$i^*\alpha = \alpha \leftarrow$ generator of H^2 dual to 2-cell

$i^*\alpha^k = \alpha^k \neq 0$

$\Rightarrow \alpha^k \neq 0$ in $H^*(\mathbb{C}P^n)$

$$\begin{array}{lccccc} \underbrace{H^*(\mathbb{C}P^n)}: & 1 & \alpha & \ldots\ldots & \alpha^{n-1} & \beta \in H^{2n} \\ & \downarrow & \downarrow & & \downarrow & \downarrow \quad \downarrow i^* \\ \underbrace{H^*(\mathbb{C}P^{n-1})}: & 1 & \alpha & \ldots\ldots & \alpha^{n-1} & 0 \end{array}$$

Need to prove: $\alpha^n = \beta$

$$(\cdot,\cdot) : H^2 \times H^{2n-2} \longrightarrow \mathbb{Z}$$

$\alpha \quad \alpha^{n-1}$ perfect pairing

$$(\alpha, \alpha^{n-1}) = \pm 1$$

$(\alpha, \alpha^{n-1}) = \pm 1$

$\alpha \cup \alpha^{n-1}([\mathbb{C}P^n]) = \alpha^n([\mathbb{C}P^n]) = \pm 1$

$0 \neq \alpha^n =$ generator of $H^{2n}(\mathbb{C}P^n)$

$\Rightarrow H^*(\mathbb{C}P^n) = \langle 1, \alpha, \ldots \alpha^n \rangle$
basis over $\mathbb{Z}$

$$= \frac{\mathbb{Z}[\alpha]}{(\alpha^{n+1} = 0)} \quad H^{2n+2} = 0$$

Thm $H^*(\mathbb{R}P^n; \mathbb{Z}_2) = \dfrac{\mathbb{Z}_2[\beta]}{(\beta^{n+1} = 0)}$

$\beta \in H^1(\mathbb{R}P^n, \mathbb{Z}_2)$

Proof: Same, just work over $\mathbb{Z}_2$, so do not need to care about orientations.

Recall that Poincaré duality works over $\mathbb{Z}_2$ for any manifold (not necessary orientable).

Complex manifold:

charts $\Leftrightarrow \mathbb{C}^n_{z_1 \ldots z_n}$

transition function = holomorphic function $\mathbb{C}^n \to \mathbb{C}^n$

Ex: polynomials / rational fns in z_i

HW: complex 1-manifold

charts $\longleftrightarrow \mathbb{C}$

transition fns $\longleftrightarrow$ holomorphic function $\mathbb{C} \to \mathbb{C}$

transition function = holomorphic function $\mathbb{C}^n \to \mathbb{C}^n$ [illegible] in z_i	transition fns $\longleftrightarrow$ holomorphic function $\mathbb{C} \to \mathbb{C}$

Fact Any complex n-manifold is an orientable real $2n$-mfd.

(In the HW, need to prove it for $n=1$)