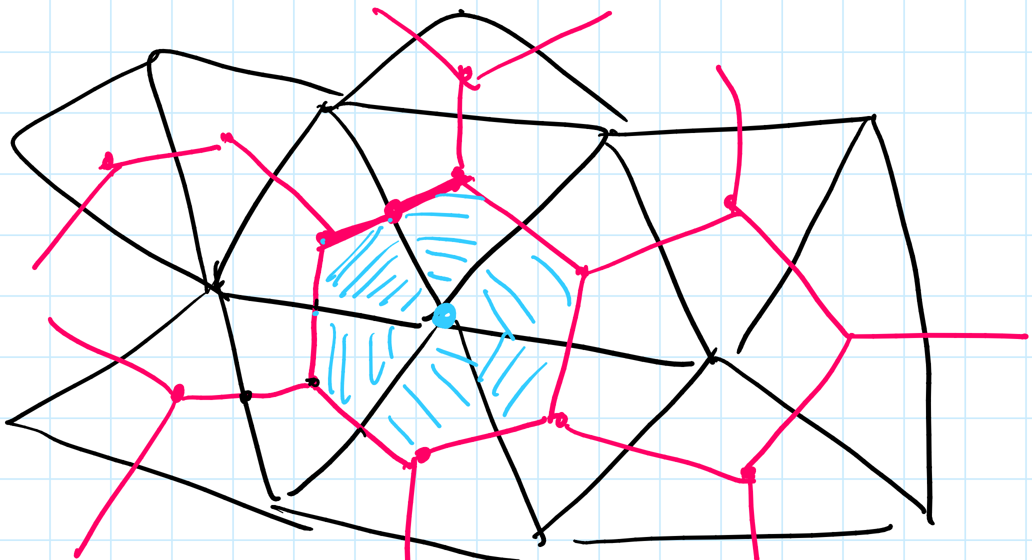
Poincaré duality, PL manifolds

M = n-dimensional oriented PL manifold



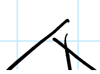
Triangulation of M $\longleftrightarrow$ dual cell decomposition

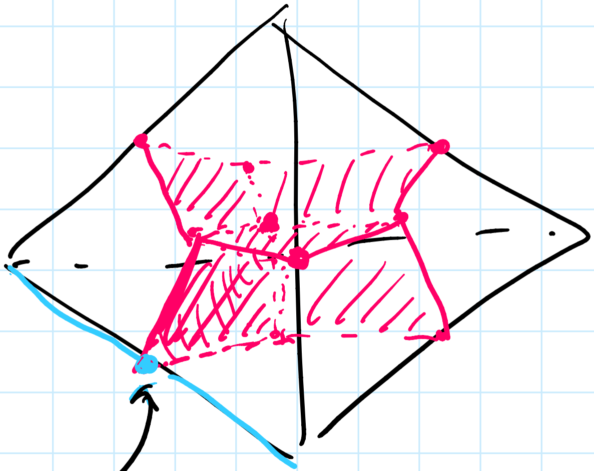
Vertices of red cell complex

$\longleftrightarrow$ n-dim simplices of the triangulation

k - cells of the cell complex

$\longleftrightarrow$ $(n-k)$ simplices of the triangulation.





1-simplex in the tetrahedron (triangulation)

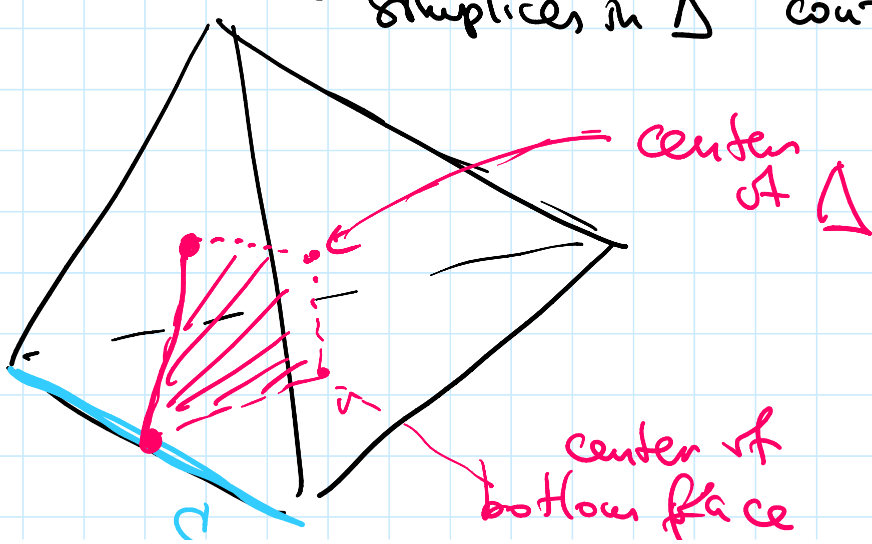
$\updownarrow$

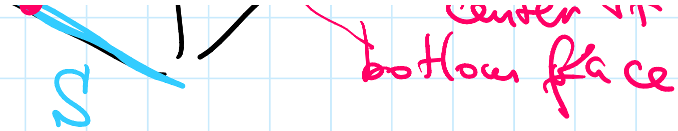
2-cell in the dual cell decomposition.

$(n-k)$-simplex = face of some n-simplex
S Δ

k - face of the dual cell decomposit.

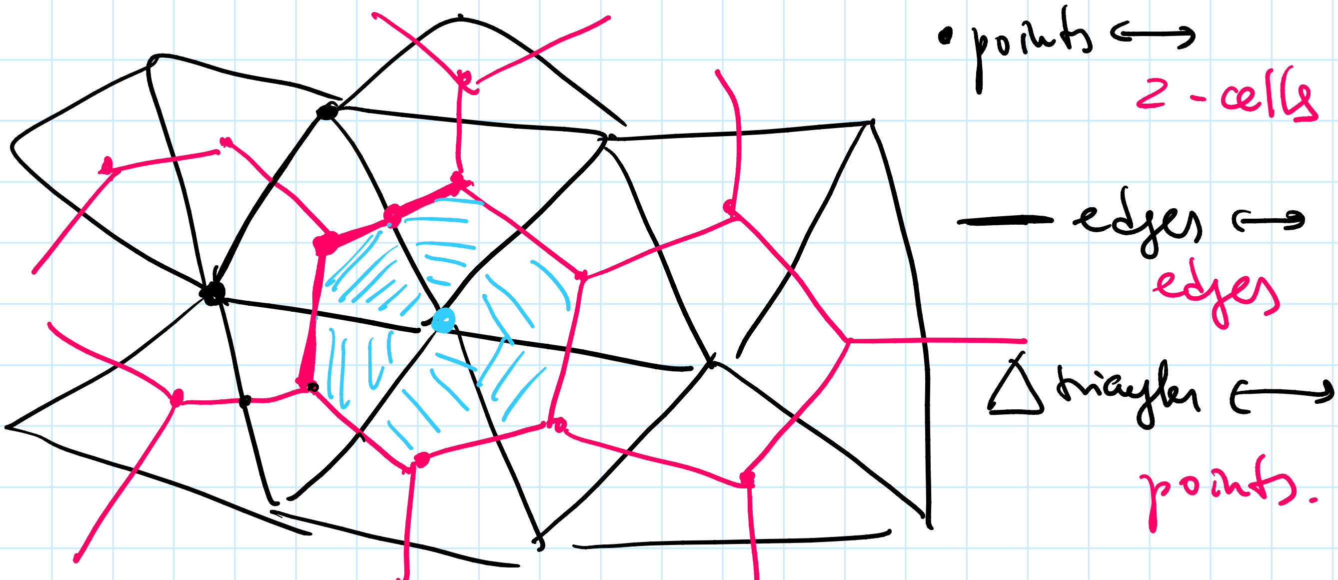
= Convex hull (baricenters of all simplices in Δ containing S)





And take union of all such k-faces over all n-simplices Δ containing S.

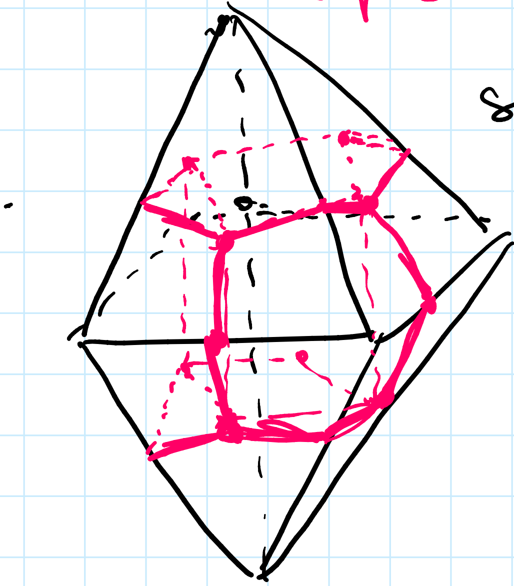
<u>Fact</u> Union of all such k-cells glues to a one k-cell.



<u>To sum up</u>: we have 2 cell decompositions of M:

- "black": simplices of the original triangulation, all boundaries are black
- "red": cells in the dual

cell decomposition, all boundaries are red.



surface of octahedron $\simeq S^2$

with 8 triangles

12 edges

6 vertices

dual cell decomposition = cube

8 Vertices $\longleftrightarrow$ centers of faces of the octahedron.

12 edges connecting these vertices on the surface.

6 faces of the cube $\longleftrightarrow$ cycles of red edges.

<u>Key idea</u>: ① We can use either of cell decompositions to compute the homology of M!

C_i = "black" i-chains =

C_i = "black" i-chains =

= Span (black i-simplices)

C_{n-i} = Span (red (n-i)-cells).

<u>Claim:</u> C_i is dual to C_{n-i}

basis of i-simplices

dual basis of cells

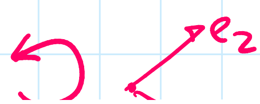
$$(\underbrace{S}_{i\text{-simplex}}, \underbrace{C}_{n-i\text{-cell}}) = \begin{cases} \pm 1 & \text{if } S \text{ intersects } C \\ 0 & \text{otherwise} \end{cases}$$

$\pm$ depends on orientation:

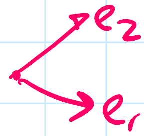
- we choose orientation for S
- we choose orientation for C
- we are given orientation of M (fixed).

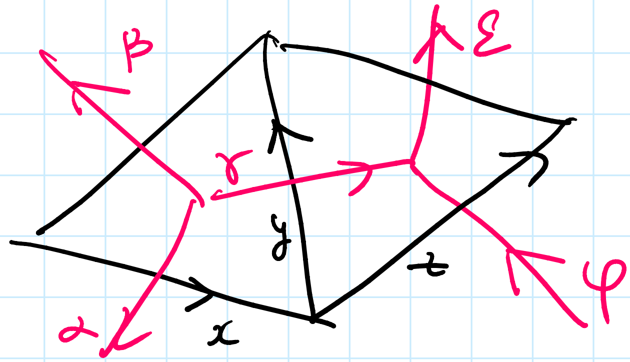
+1 if (basis for S) $\sqcup$ (basis for C) has same orientation as M

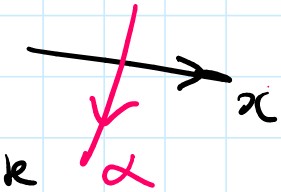
-1 if opposite.

<u>Ex:</u> M 

orientation go from e_1 to e_2 counter-clockwise

Ex: M ↺  orientation: go from e_1 to e_2 counter-clockwise

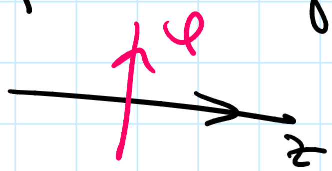


(x, α) 

has opposite orientation from M
$\Rightarrow -1$

$(x, \beta) = 0 \qquad (x, \gamma) = 0$

$(z, \varphi) =$ agrees with orientation of $M \Rightarrow +1$.



$(z, \alpha) = 0$ extend to a bilinear pairing.

$$(3x + 2y + 5z, \alpha + \beta + \varphi) =$$

$$= 3(x, \alpha) + 5(z, \varphi) \quad (\text{all other} = 0)$$

$$= -3 + 5 = 2.$$

We get the following picture:

$$C_i \xrightarrow{\partial} C_{i-1}$$

C_i ↕ dual $\qquad$?

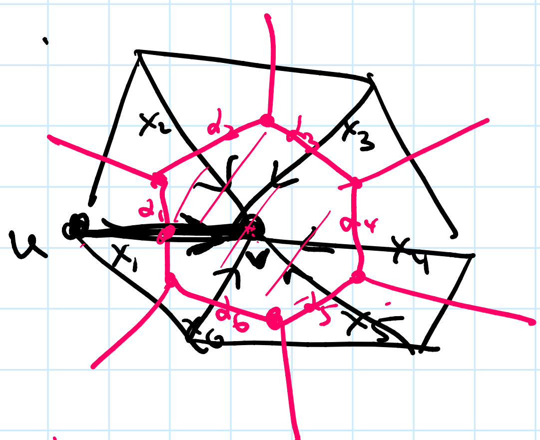
$$\begin{tikzcd} C_{n-i} \arrow[r, "\partial"] & C_{n-i-1} \\ C^i \arrow[u, phantom, "\cong"] \arrow[r, "\delta"] & C^{i+1} \arrow[u, phantom, "\cong"] \end{tikzcd}$$

dual

?

= dual to C_{i+1}

Claim: The differential on red cell complex is dual to the differential for black triangulation



F = red 2d face

$$\partial F = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$$

$$(\partial F, x_1) = (\alpha_1, x_1) = \pm 1$$

$$\|$$

$$(F, \partial x_1) = (F, v) = \pm 1$$

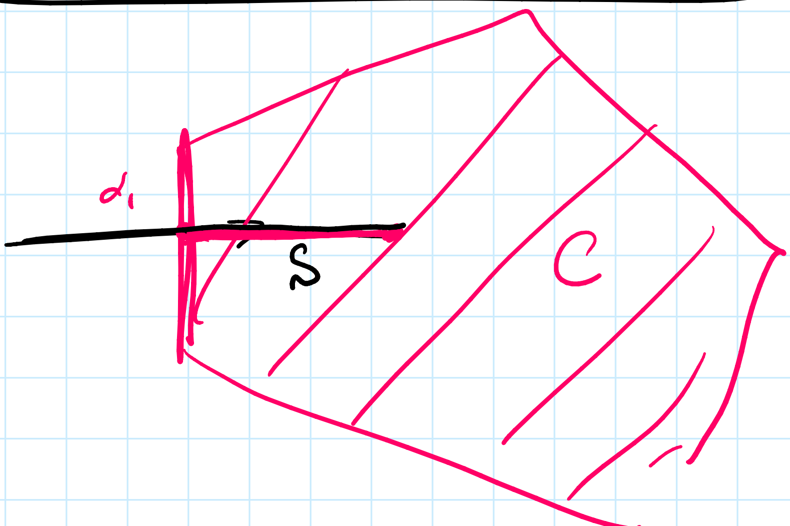
$$\partial x_1 = v - u$$

S = black i simplex

C = red cell $n-i+1$

simplex

$$\boxed{(\underset{i-1}{\partial S}, \underset{n-i+1}{C}) = (\underset{i}{S}, \underset{n-i}{\partial C})} \quad (**)$$



Why this is useful:

$$C_{n-i} = C^i = (C_i)^*$$

(**) means that the differential in C_{n-i} is dual to the differential in C_i = same as differential in C^i

$$H_{n-i}(M) = H_\cdot(C_{n-i}, \partial) =$$

$$= H^\cdot(C^i, \delta) = H^i(M)$$

We know that homology and

We know that homology and cohomology computed using different cell decompositions are isomorphic

So $\boxed{H_{n-i}(M) \simeq H^i(M) \quad \text{for all } i}$