Lecture 16 (5/3)

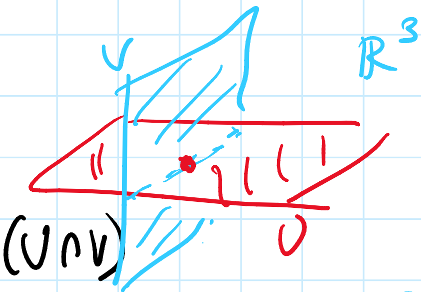
Monday, May 3, 2021 2:10 PM

Detour: Transversality

Linear algebra:

$U^k, V^l \subset \mathbb{R}^n$ subspaces

We say that U and V are transversal if $U + V = \mathbb{R}^n$



$$\dim(\underset{n}{U+V}) = \dim \underset{k}{U} + \dim \underset{l}{V} - \dim(U \cap V)$$

$U+V=\mathbb{R}^3$

$\dim U = 2 = \dim V$

$\dim U \cap V = 1$

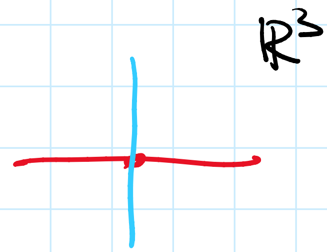
$\Rightarrow \dim(U \cap V) = k + l - n.$

Dim $(U \cap V)$ is minimal possible.

• If $k+l < n$, never transversal

• If $k+l=n$, U and V are transversal if $U \cap V = 0 \iff$

$$U \oplus V = \mathbb{R}^n.$$



• $\operatorname{codim}(U) = n - \dim U = n - k$

$\operatorname{codim}(V) = n - l$

$\operatorname{codim}(U \cap V) = n - (k+l-n) =$

$(n-k)+(n-\ell)$

$$\boxed{\begin{gathered}\operatorname{codim}(U\cap V)=\operatorname{codim}U+\operatorname{codim}V\\ \text{if } U \text{ is transversal to } V\end{gathered}}$$

$M = n$-dimensional <u>smooth</u> manifold

$N^k, L^\ell =$ two smooth submanifolds

(locally N looks like $\mathbb{R}^k \subset \mathbb{R}^n$

L looks like $\mathbb{R}^\ell \subset \mathbb{R}^n$)

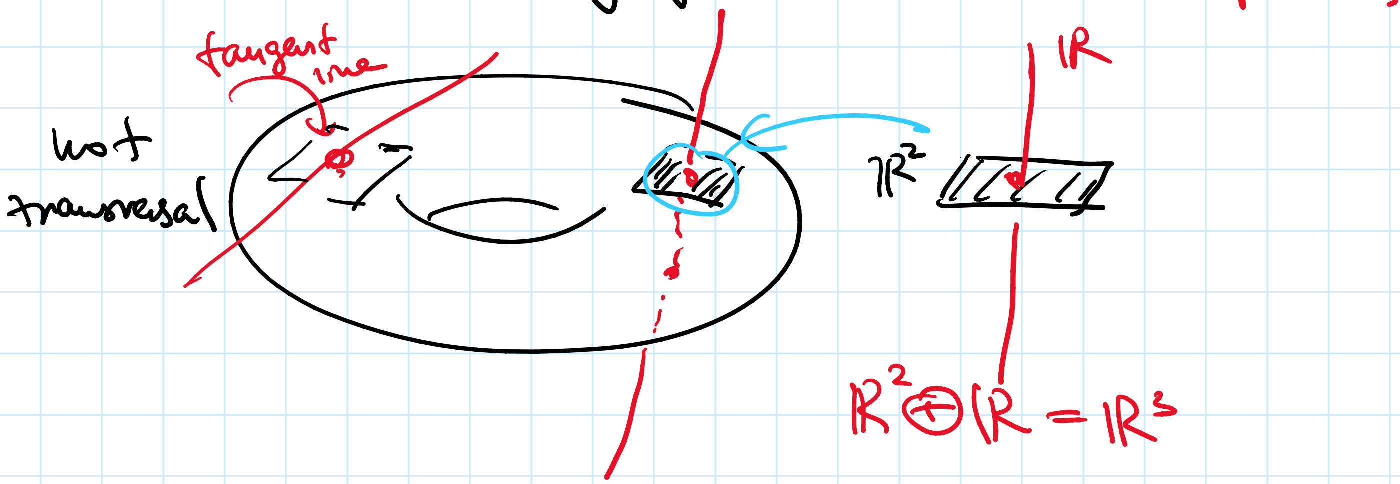
<u>Def</u> N and L are transversal in M if they are transversal at every point of their intersection $N \cap L$ — that is, at $N \cap L$ they locally look like (up to change of coords) transversal subspaces

two 2-planes, intersect transversally.

<u>Rmk</u> Precisely: $T_pN + T_pL = T_pM$

Rmk Precisely: $T_pN + T_pL = T_pM$ — tangent spaces.

for every $p \in L \cap L$

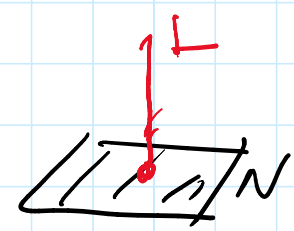


Facts: ① If $k+l<n$ then

N is transverse to $L \iff N \cap L = \emptyset$

② If $k+l=n$ then $N \cap L$

is a discrete set of points where



N and L locally look like transverse

vector spaces (no tangencies).

③ If $k+l \geq n$ and N is transverse

to L then $N \cap L$ is a submanifold of M

of dimension $k+l-n$.

$$\boxed{\operatorname{Codim}(N \cap L) = \operatorname{Codim} N + \operatorname{Codim} L}$$

$$\boxed{\operatorname{codim}(N \cap L) = \operatorname{codim} N + \operatorname{codim} L}$$

as in linear algebra.

④ (Thom) transversality theorem:

M = smooth n-dim manifold

N^k, L^ℓ = smooth submanifolds

$\Longrightarrow$ one can "perturb" N and L to make them transversal.

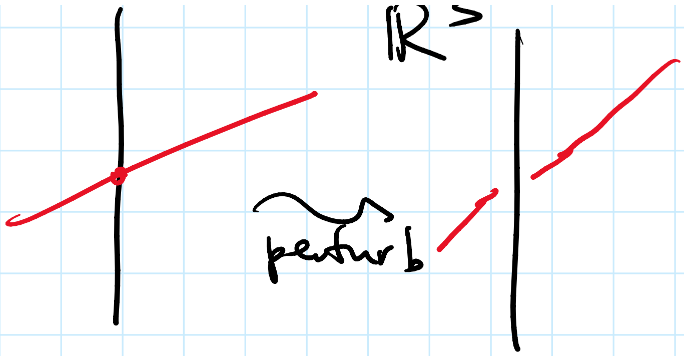
≈ general position

locally ~ shift by some vector.

Idea: Sard's theorem on regular values. (239).

Ex N^k, L^ℓ = any submanifolds of M and $k+\ell < n \Rightarrow$ by ④ we can perturb them so that they do not intersect.

$\mathbb{R}^3$



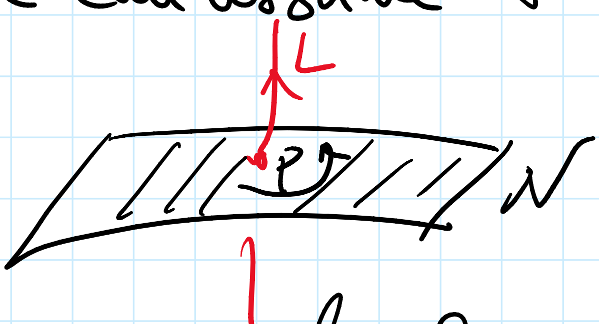
How does it help us understand the <u>Poincaré duality</u>/cup product?

M = compact, oriented n-manifold

$N^k \subset M$ = smooth oriented submanifold

$L^{n-k} \subset M$ = smooth oriented submanifold

By Thom transversality theorem, we can assume that $N \cap L$ = finitely many points



At each intersection point p, N has a local orientation and L has a local orientation.

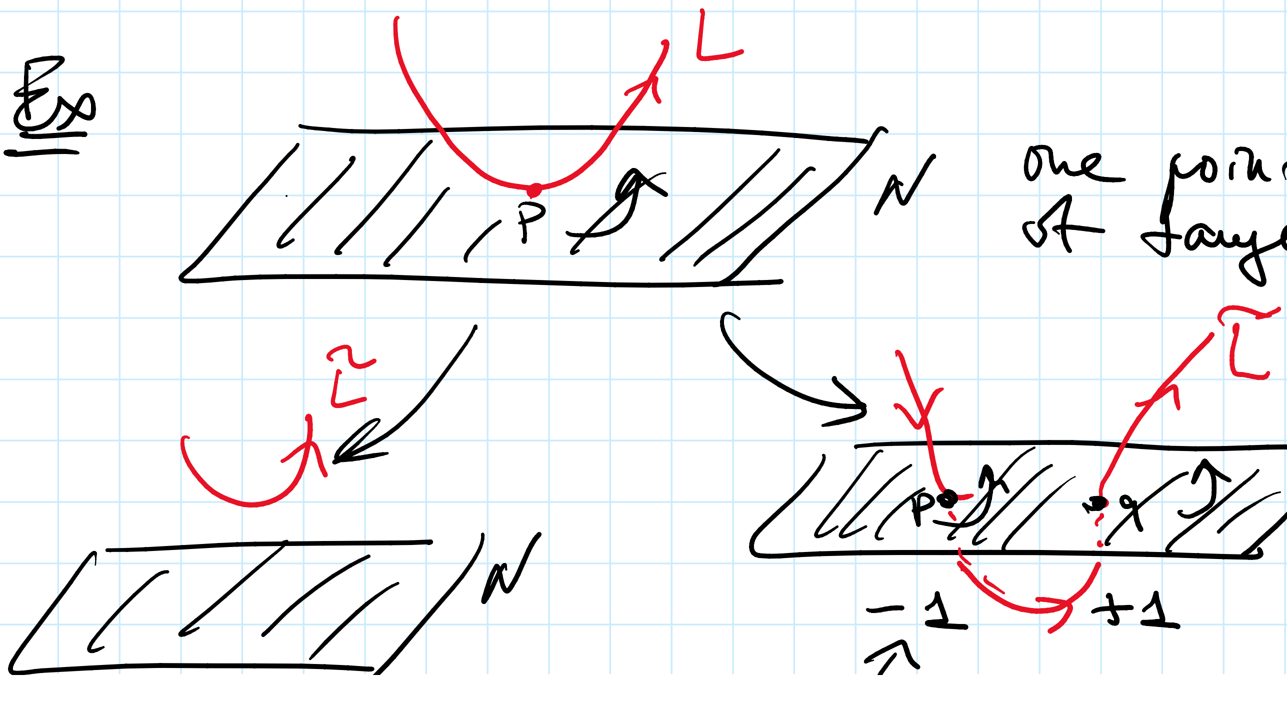
Intersection index:

local at $p \in N \cap L$ $\begin{cases} +1, \text{ if (orientation of } N) \cup \text{(orientation of } L) = \text{orientation of } M \\ -1, \text{ if they have opposite orientation.} \end{cases}$

Then add up these local intersection indices over all intersection points.

Thm This is well defined, does not depend on perturbation of N and L and depend only on homology classes $[N] \in H_k(M)$ $[L] \in H_{n-k}(M)$.

Ex



one point of tangency

$L \cap N = \emptyset$

$\Rightarrow$ intersection index $= 0$

$-1 \quad +1$

(orientation on N) $\sqcup$ (orientation on L)

$\neq$ (orientation on $\mathbb{R}^3$).

$(x, y, -z)$

$(+1) + (-1) = 0$

<u>Notations</u>: $N \circ L$ = intersection index

<u>Remk</u> $N \circ L \neq L \circ N$!

$$\boxed{N \circ L = (-1)^{kl} L \circ N}$$

(same sign rule again!)

$$\underbrace{\begin{pmatrix}\text{basis in} \\ \text{local model} \\ \text{for } N\end{pmatrix} \sqcup \begin{pmatrix}\text{basis in} \\ \text{local} \\ \text{model for } L\end{pmatrix}}\ \longleftrightarrow \ (\text{swap bases})$$

kl transpositions.

<u>Thm</u> This is related to Poincaré duality!

fix N^k, let L^l vary. $\quad l = n-k$

$\phi_N : H_l \longrightarrow \mathbb{Z}$

$\phi_N([L]) \longmapsto N \circ L$

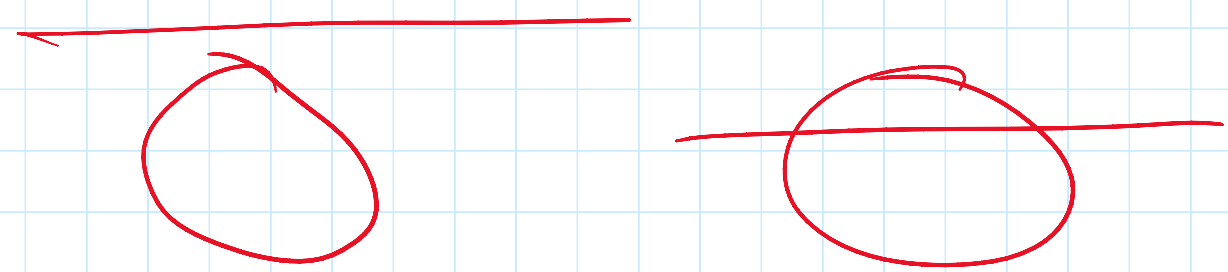
Assuming all cycles are realized by submanifolds

$\rightsquigarrow$ N defines a function on $H_{n-k}(M)$

$\longleftrightarrow$ an element in $H^{n-k}(M)$.

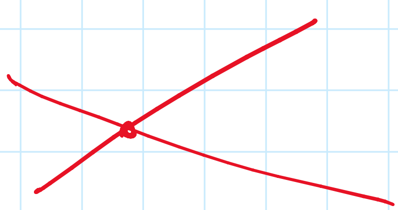
$$H_k \longrightarrow H^{n-k}(M)$$

<u>This is the same map as prescribed by Poincaré duality.</u>



$\underline{\mathbb{CP}^1} \subset \mathbb{CP}^2$

$[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2)$



$\mathbb{CP}^2$

$[\mathbb{CP}^1] \cdot [\mathbb{CP}^1] = 1$

$[\mathbb{CP}^1]$ defines a function

$H_2(\mathbb{CP}^2) \longrightarrow \mathbb{Z}$

which sends $\underline{[\mathbb{CP}^1] \longrightarrow 1}$.

$\alpha \in H^2(\mathbb{CP}^2)$ = generator of H^2

$\alpha \cup \alpha$ = generator in H^4.