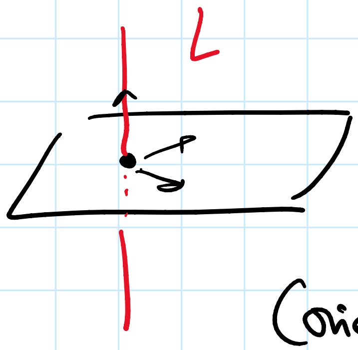
Lecture 17 (5/5)

Wednesday, May 5, 2021 2:09 PM

M^n = smooth orientable manifold

N^k, L^{n-k} (oriented) transversal



At every point of intersection we have

(orientation of N) $\cup$ (orientation of L)

vs

(orientation of M) $\rightsquigarrow \pm 1$

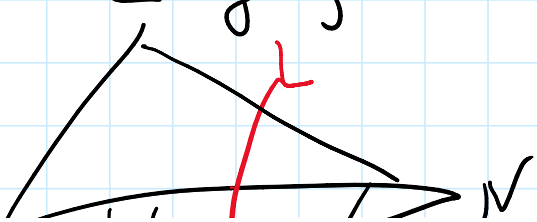
$N \cdot L$ = intersection index = sum of all these local intersection indices

Can define the same thing if

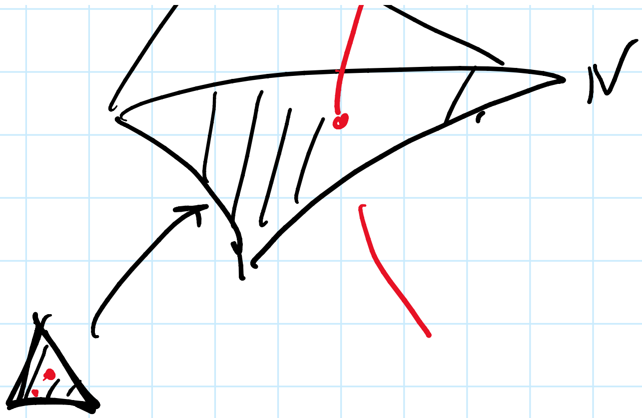
N, L = singular chains.

$N = \sum a_i \sigma_i \qquad \sigma_i : \Delta^k \longrightarrow M$ singular simplices

$L = \sum b_j \omega_j \qquad \omega_j : \Delta^{n-k} \longrightarrow M$



We can assume σ_i, ω_j smooth

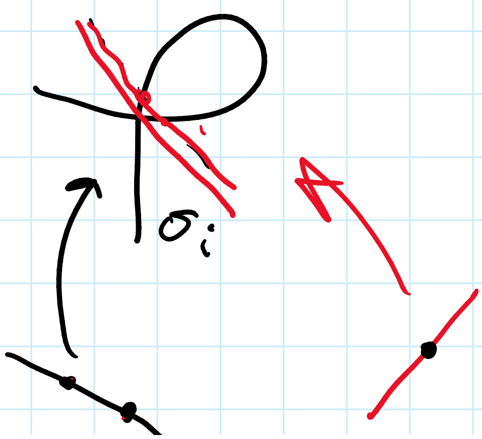


σ_i, ω_j smooth and each σ_i intersects ω_j transversally at interior points.

$$\sigma_i \cap \omega_j = \sigma_i^{-1}\Big(\underbrace{\omega_j\left(\Delta^{n-k}\right)}_{\substack{\text{singular}\\ (n-k)\text{-simplex}\\ \text{in } M}}\Big)$$

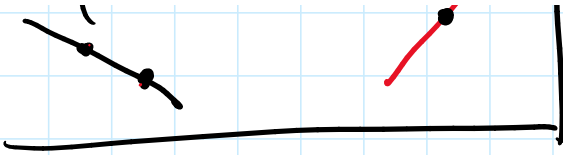
By slightly stronger transversality thm, we can assume that $\sigma_i^{-1}(\omega_j(\Delta^{n-k}))$ is a collection of points at the interior of Δ^k.

<u>Transversality</u>: $D\sigma_i(T\Delta^k)$ is transverse to $D\omega_j(T\Delta^{n-k})$ in TM



$$\boxed{N \circ L = \sum_{i,j} a_i b_j (\sigma_i \cdot \omega_j)}$$

intersection



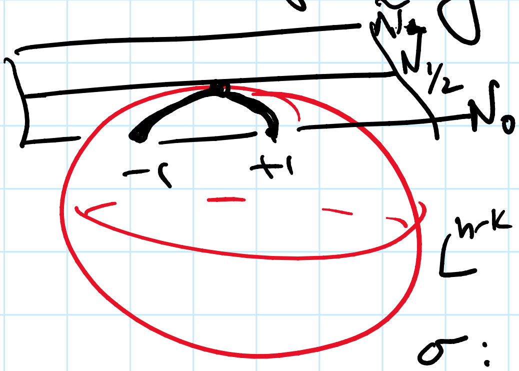
intersection index computed by the above procedure.

Key question:

① Why this does not depend on perturbation?

② Why this only depends on homology classes of N and L?

Idea: ① Suppose that we have a family of perturbations.

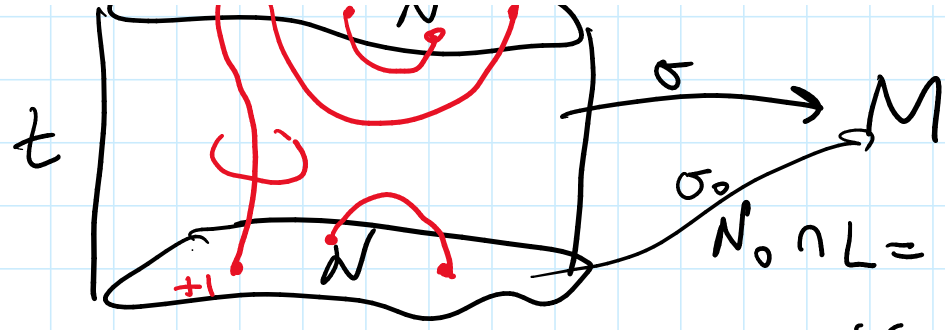


we can think of it as a map

$$\sigma : N \times [0,1] \longrightarrow L$$

$$N_t \xrightarrow{t} M$$

Observe: If N was smooth, $N \times [0,1]$ is also smooth, of dimension $k+1$



$N_0 \cap L = \sigma_0^{-1}(L)$

$= \sigma^{-1}(L) \cap \{t=0\}$

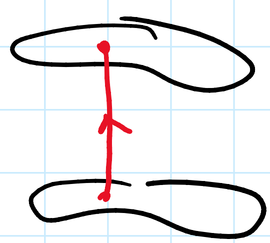
By Thom transversality, $\sigma^{-1}(L)$ is a smooth compact 1-submanifold Z of $N \times [0,1]$

$Z \cap \{t=0\} = N_0 \cap L$ with signs

$Z \cap \{t=1\} = N_1 \cap L$ with signs.

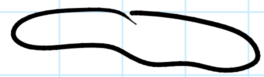
Z has the following components:

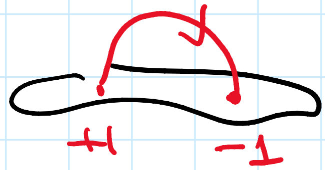
- Segments connecting points on the bottom with points on the top



- Segments connecting points on the bottom with points on the bottom

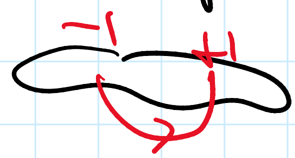
bottom

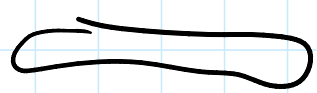




signs cancel out
$(+1)+(-1)=0$

- Segments connecting points on the top & points on the top





- Closed circles — do not contribute.

$\Rightarrow$ Σ points on the bottom with signs $=$

Σ points on the top with signs.

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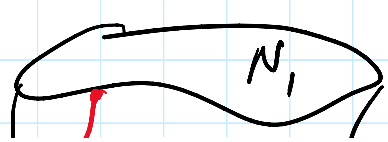
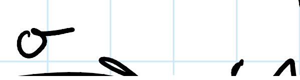
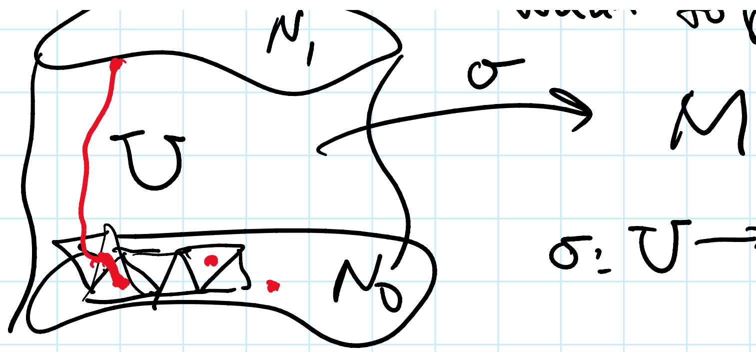
② If $\underline{N_0} - \underline{N_1} = \partial$ (some singular chain U)

Want to prove $N_0 \cdot L = N_1 \cdot L$





want: to prove $N_0 \circ L = N_1 \circ L$



$\sigma : U \to M$

$$\sigma(N_0) \cap L \leftrightarrow \sigma^{-1}(L) \cap N_0$$

Can assume that on N_0 and N_1 all intersection points are isolated, not on the boundary of simplices, (in N_0) and $\underline{\sigma^{-1}(L) \subset U \text{ is a PL } 1\text{-manifold}}$ and we can use the previous argument.

<u>Summary</u>: We have a well defined pairing $H_k(M) \times H_{n-k}(M) \to \mathbb{Z}$

$$N, L \longrightarrow N \circ L$$

intersection index

This is related to Poincaré duality:

This is related to Poincaré duality.

$$H_k(M) \longrightarrow H^{n-k}(M) = \{\text{functions on } H_{n-k}(M)\}$$

$$N \longrightarrow N \cdot \_\_$$

This is the most concrete description of Poincaré duality (up to torsion)

$$\underset{H_k(M)}{\overset{N}{\cap}} \longrightarrow \underset{H^{n-k}(M)}{\overset{PD(N)}{\cap}} \quad \text{defined by}$$

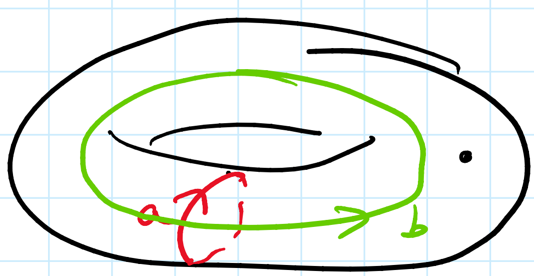
$$N \cdot L = PD(N)(L)$$

$$L \in H_{n-k}(M)$$

<u>Claim:</u> $\underset{H^{n-k}}{PD(N)} \cup \underset{H^k}{PD(L)} \left([M]\right) =$

$$= N \cdot L$$

<u>Ex</u> T^2



$H_0 = \langle pt \rangle$

$H_1 = \langle a, b \rangle$

$H_2 = [T^2]$

Intersection form:

$(pt) \cdot [T^2] =$ one point with positive orientation.

$[T^2] \cdot [pt]$

$[T^2]\cdot[pt]$ with positive orientation.

$\shortparallel$

$$[pt]\cdot[T^2] = +1$$

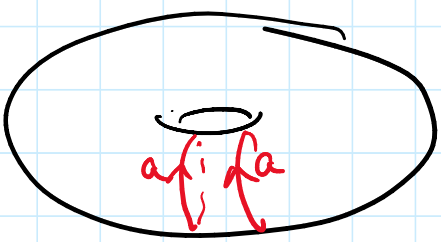
$H_0 \ni [pt] \longrightarrow$ class $PD[pt] \in H^2(T^2)$ generator

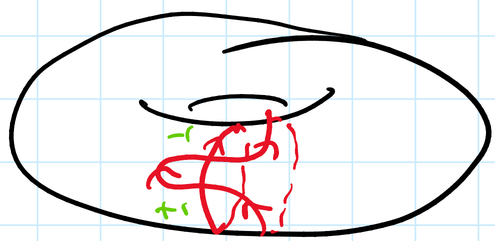
characterized by $PD[pt]([T^2]) = +1$

$H_2 \ni [T^2] \longrightarrow$ class $PD[T^2] \in H^0(T^2)$

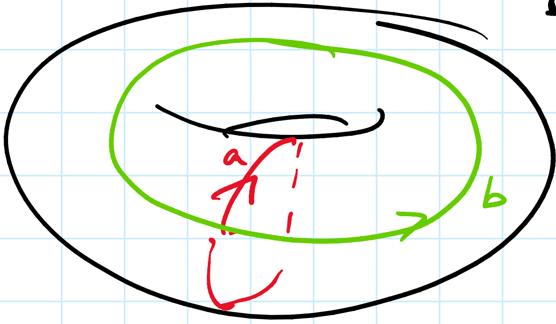
value on $pt = 1$

$a \cdot a = 0$ ← cannot just take $a \cap a$, need to perturb





$$\left.\begin{aligned} a\cdot b &= -1 \\ b\cdot a &= 1 \end{aligned}\right\}$$ transverse intersection do not need to perturb



$b \cdot b = 0$.

This gives a complete description of the intersection form on H_1, can use this to describe Poincaré duality:

To describe Poincaré duality:

$$\alpha = PD[a] \qquad \beta = PD(b) \quad \in H^1(T^2)$$

$$\alpha(a) = a \cdot a = 0 \qquad \alpha(b) = a \cdot b = -1$$

$$\beta(a) = b \cdot a = 1 \qquad \beta(b) = b \cdot b = 0.$$

$$\alpha \cup \beta\,([T^2]) = a \cdot b = -1$$

$$\boxed{\alpha \cup \beta = -PD(pt)}$$

HW#4: do the same for genus g surface