Lecture 18 (5/7)

Friday, May 7, 2021 2:10 PM

M = smooth oriented n-manifold

N^k, L^ℓ = smooth oriented submanifolds

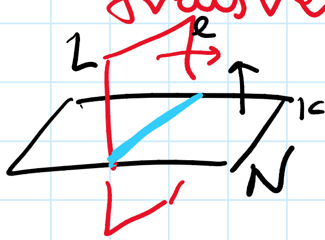
Assume N and L are transverse

(if they are not, by Thom transversality theorem perturb to make them transverse)

$N \cap L$ = smooth submanifold

follows from transversality

of dimension $k+\ell-n$



(empty if $k+\ell<n$)

If N is oriented, L is oriented can orient $N \cap L$ canonically as well.

Fix orientation of M, then locally

(orientation of N) $\Leftrightarrow$ (orientation of $N^\perp$)

locally N = k-space

(orientation of L) $\Leftrightarrow$ (orientation of $L^\perp$)

$(L \cap N)^\perp = L^\perp \oplus N^\perp$ if L and N

$(L \cap N)^{\perp} = \underline{L^{\perp} \oplus N^{\perp}}$ if L and N are transverse.

$(\text{basis in } N) \cup (\text{basis in } N^{\perp}_{j}) =$

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det A \cdot \det B$$

Knowing two of these dets yields the third.

← orientation of $N^{\perp}$

orientation of $\mathbb{R}^3$ (right hand rule)

$\Longrightarrow$ orientation of surface

<u>Note:</u> If $k+\ell = n$, $N \cap L$ is a collection of points, and orientation on $N \cap L \Leftrightarrow$ signs of intersections at these points.

$$[N] \in H_k(M) \xleftrightarrow{PD} \alpha \in H^{n-k}(M)$$

$$[L] \in H_\ell(M) \xleftrightarrow{PD} \beta \in H^{n-\ell}(M)$$

<u>Fact</u> $[N \cap L] \in H_{k+\ell-n}(M)$

$\Updownarrow PD$

$\alpha \wedge \beta \in H^{2n-k-\ell}(M)$ ...

class in $H^{2n-k-\ell}(M) \overset{!}{=} \alpha \cup \beta$ $\updownarrow PD$

<u>Idea of proof</u>: Recall that

$\Delta : M \longrightarrow M \times M$ diagonal

$\alpha \cup \beta = \Delta^*(\alpha \otimes \beta)$

After using Poincaré duality,

$\Delta^*(\alpha \otimes \beta)$ is Poincaré dual

to intersection of $N \times L \subset M \times M$
with the diagonal

Key idea: $N \times L \cap \Delta =$
$= \{(a,b) : a \in N, b \in L\} = N \cap L$ ($a = b$)

Let us apply this fact

to study H_* and H^* of $\mathbb{CP}^n$

<u>We know:</u>

- $\mathbb{CP}^n$ has a cell decomposition

$0, 2, 4, \ldots 2n$ — cells $\Rightarrow$

$\Rightarrow H_0 = H_2 = \ldots = H_{2n} = \mathbb{Z}$

$H^0 \quad H^2 \quad \quad H^{2n} = \mathbb{Z}$

$= H_0 = H_2 = \dots = H_{2n} = \mathbb{Z}$

$H^0 = H^2 = \dots = H^{2n} = \mathbb{Z}$

Odd homology $= 0$.

- $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$ hyperplane

<u>Reminder</u>: $\mathbb{C}P^n = \dfrac{\mathbb{C}^{n+1} \setminus \{0 \dots 0\}}{\mathbb{C}^*}$

Homogeneous coordinates:

$[x_0 : \dots : x_n]$

$\sim$

$[\lambda x_0 : \dots : \lambda x_n]$ $\quad \lambda \neq 0$

$H = \{x_0 = 0\}$ invariant under action of λ

$H \simeq \mathbb{C}P^{n-1}$ with coords $[0 : x_1 : \dots : x_n]$

smooth oriented

$[H]$ = generator of $H_{2n-2}(\mathbb{C}P^n)$

(follows from cell decomposition)

PD

$a \in H^2(\mathbb{C}P^n)$ $\quad$ a is a generator of H^2

"hyperplane class"

$a^2 = a \cup a \xleftrightarrow{PD} [H \cap \tilde{H}]$

perturb H

$\tilde{H} = \{\gamma_0 z_0 + \dots + \gamma_n z_n = 0\}$ $\quad \mathbb{C}^n \subset \mathbb{C}^{n+1}$

$\shortparallel$

$\mathbb{C}P^{n-1}$

$\tilde{H} = \{\gamma_0 z_0 + \dots + \gamma_n z_n = 0\}$ ← $\mathbb{C}^n \subset \mathbb{C}^{n+1}$

$\;\;\|$

$\mathbb{CP}^{n-1}$ ← generic linear equation

<u>Note:</u> $\gamma_0(\lambda z_0) + \dots + \gamma_n(\lambda z_n) = \lambda(\gamma_0 z_0 + \dots + \gamma_n z_n) \Rightarrow \tilde{H}$ is invariant under action of λ

Same is true for any homogeneous equation.

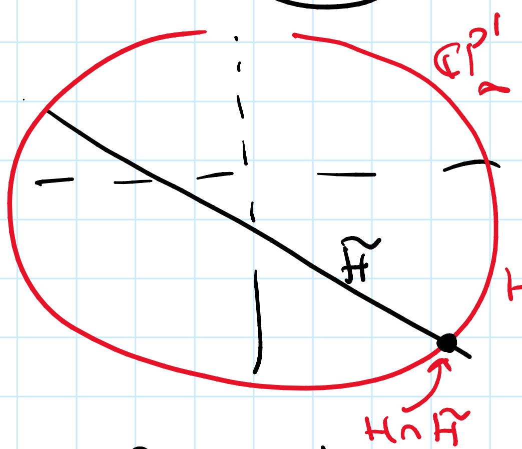
For any choice of γ_i, $[\tilde{H}] = [H]$

$$[H \cap \tilde{H}] = \begin{Bmatrix} z_0 = 0 \\ \gamma_0 z_0 + \dots + \gamma_n z_n = 0 \end{Bmatrix} =$$

(complex) codim 2 subspace $= \mathbb{CP}^{n-2}$

$$\begin{array}{ccc} \mathbb{C}^3 \setminus 0 \ [x_0 : x_1 : x_2] & \longleftarrow & \mathbb{C}^2 \\ \downarrow & x_0 + 2x_1 + 3x_2 = 0 & \downarrow \\ \mathbb{CP}^2 & \ell \ \supset & \mathbb{CP}^1 \end{array}$$

<u>Chart:</u> $x_0 \neq 0$ $\quad 2x_1 + 3x_2 + 1 = 0$



$\mathbb{C}^2$

in this chart we divide by x_0 and assume $x_0 = 1$

chart $\simeq \mathbb{C}^2(x_1, x_2)$

Infinite line: $x_0 = 0$

Inf point line: $x_0 = 0$

$$\mathbb{CP}^1_\infty \cap \ell = \begin{cases} x_0 + 2x_1 + 3x_2 = 0 \\ x_0 = 0 \end{cases}$$

$$\Longleftrightarrow \begin{array}{l} x_0 = 0 \\ 2x_1 + 3x_2 = 0 \end{array}$$

One point $[0 : 1 : -\frac{2}{3}]$

Conclusion: $a^2 \xleftrightarrow{PD} [\mathbb{CP}^{n-2}]$ codim 2 subspace

And we can continue:

$$a^3 \xleftrightarrow{PD} \left[\begin{array}{c}\text{intersection of} \\ 3 \text{ hyperplanes}\end{array}\right] \simeq [\mathbb{CP}^{n-3}]$$

$$\cdots\cdots$$

$$a^{n-1} \xleftrightarrow{PD} \left[\begin{array}{c}\text{intersection of} \\ n-1 \text{ hyperplanes} \\ \text{in } \mathbb{CP}^n\end{array}\right] \simeq [\mathbb{CP}^1] = \underline{\underline{\text{line}}}$$

$$a^n \xleftrightarrow{PD} \left[\begin{array}{c}\text{intersection of} \\ n \text{ hyperplanes}\end{array}\right] \simeq [pt] = pt.$$

$1, a, a^2 \ldots a^n$

$[\mathbb{CP}^n]\ [\mathbb{CP}^{n-1}] \ldots [pt].$

Application:

Application:

$$Q := \{x_0^2 + \dots + x_n^2 = 0\}$$

homogeneous quadric

$$(\lambda x_0)^2 + \dots (\lambda x_n)^2 = 0$$
$$\parallel$$
$$\lambda^2 (x_0^2 + \dots + x_n^2)$$

Fact Q is a smooth complex submanifold of complex dim $n-1$ of $\mathbb{C}P^n$

$$\Rightarrow [Q] \subset H_{2n-2}(\mathbb{C}P^n)$$

Question: compute this class.

Answer: Poincaré duality!

$$PD[Q] = \text{some multiple of } a \in H^2$$

$$Q \cap (\text{line}) \xleftrightarrow{PD} PD[Q] \cup PD[\text{line}]$$
$$\parallel$$
$$PD[Q] \cdot a^{n-1}$$

$$Q = \{x_0^2 + \dots + x_n^2 = 0\}$$

$$\text{line} = [x_0 : x_1 : 0 \dots 0]$$

$$x_0^2 + x_1^2 = 0$$

[illegible] $x_0^2 +$ [illegible] $x_0 x_1 +$ [illegible] $x_1^2 =$ [illegible]

$x_0^2 + x_1^2 = 0$ $\quad$ $\underline{a x_0^2 + b x_0 x_1 + c x_1^2 = 0}$

$\Rightarrow x_1 = \pm\sqrt{-1}\, x_0$

$x_0 \neq 0 \Rightarrow$

two points $[1 : \sqrt{-1}]$

$[1 : -\sqrt{-1}]$

$Q \cap \text{line} = 2$ points!

$$PD[Q] \cdot a^{n-1} = PD[2 \text{ points}] = 2a^n$$

$$\Rightarrow PD[Q] = 2a$$

$$[Q] = 2[H]$$

__Fact__ M^n = complex manifold

automatically oriented

$\underline{N}^k, \underline{L}^{n-k}$ = complex submanifolds

$\Rightarrow$ all points of intersection have __positive__ orientation

$z_1 \dots z_n$ = local coord

$(x_1, y_1) \dots (x_n, y_n)$ $\quad$ any perm of z_i

$\uparrow$

permutation of pairs $\Rightarrow$ even.

permutation of pairs $\Rightarrow$ even.

Rmk Can do same computation in $H_*(\mathbb{RP}^n, \mathbb{Z}_2)$

With coefficients in $\mathbb{Z}_2$, orientations do not matter and we compute intersection indices mod 2.

Rmk $\{$all quadrics$\}$

$$\| $$

$$\left\{ \underbrace{n \times n \text{ square matrices}}_{\{\sum c_{ij} x_i x_j = 0\}} \right\}$$

Smooth if $\underbrace{\det(c_{ij}) \neq 0}$

complex codim 1 in the space of all matrices

real codim 2

$c_{00} x_0^2 + c_{01} x_0 x_1 + c_{11} x_1^2$

$c_{00} + c_{01} \frac{x_1}{x_0} + c_{11} \left(\frac{x_1}{x_0}\right)^2$

quadratic eqn for $\frac{x_1}{x_0}$

$\Rightarrow$ two solutions

$\{$all polynomials of deg $d\}$

$$\cup$$

$\{$non-smooth = algebraic condition on coefs$\}$

complex codim ≥ 1

Cor: any two smooth deg d hypersurfaces are deg a hypersurfaces are homeomorphic. (complex codim ≥ 1 $\Rightarrow$ real codim ≥ 2.