$$\cdots \to C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} C_{i-1} \to \cdots$$

chain complex

$$\cdots \leftarrow C^{i+1} \xleftarrow{\delta} C^{i} \xleftarrow{\delta} C^{i-1} \leftarrow \cdots$$

dual complex

Lemma (a) Suppose that α is a cocycle $\delta\alpha = 0$, then α yields a well defined function on H_i.

(b) If α is a coboundary, $\alpha = \delta\beta$ then this function on H_i vanishes.

Proof: (a) $H_i = \dfrac{\operatorname{Ker}\partial}{\operatorname{Im}\partial}$ $\quad x \in C_i \quad \partial x = 0$

$x \sim x + \partial y, \ y \in C_{i+1}$

$$\alpha(x+\partial y) = \alpha(x) + \alpha(\partial y) =$$

$$= \alpha(x) + \cancel{(\delta\alpha)(y)} \to 0$$

since $\delta\alpha = 0$.

(b) $\alpha(x) = (\delta\beta)(x) = \beta(\partial x) = 0.$

$\partial x = 0$ since x is a cycle

Cor We have a map $H^i \longrightarrow (H_i)^*$

$\alpha \longrightarrow$ corresponding function on H_i

Thm If $C_\bullet$ = complex of f.d. vector spaces over a field, then the map $H^i \longrightarrow (H_i)^*$ is an isomorphism

Cor In this case $\dim H^i = \dim H_i$,

Proof Algebraic fact: any complex of f.d. vector spaces = direct sum of

$$0 \longrightarrow \mathbb{K} \longrightarrow 0 \quad (*) \qquad 0 \longrightarrow \mathbb{K} \xrightarrow{1} \mathbb{K} \longrightarrow 0 \quad (**)$$

Need to check thm for these!

$$(0 \to \mathbb{K} \to 0)^* = 0 \to \mathbb{K} \to 0$$

$$(0 \to \mathbb{K} \to \mathbb{K} \to 0)^* = 0 \to \mathbb{K} \xrightarrow{1} \mathbb{K} \to 0$$

H_* counts the number of summands of type $(*)$, Same for H^*. $\boxtimes$

HW #2 : given a different proof by computing $\dim \operatorname{Ker}$, $\dim \operatorname{Im}$ for ∂, δ.

$$\to A_{i+1} \xrightarrow{\partial_A} A_i \xrightarrow{\partial_A} A_{i-1} \to \cdots$$

$$\to B_{i+1} \xrightarrow{\partial_B} B_i \xrightarrow{\partial_B} B_{i-1} \to \cdots$$

$$\to A_{i+1} \oplus B_{i+1} \xrightarrow{\partial_A \oplus \partial_B} A_i \oplus B_i \to A_{i-1} \oplus B_{i-1} \to \cdots$$

$$\boxed{\begin{array}{c|c} \partial_A & 0 \\ \hline 0 & \partial_B \end{array}} \qquad H_*(A_\bullet \oplus B_\bullet) = $$

$$= H_*(A_\bullet) \oplus H_*(B_\bullet)$$

$$(A_\bullet \oplus B_\bullet)^* = A_\bullet^* \oplus B_\bullet^*$$

<u>Thm</u> If $C_\bullet$ complex of free abelian groups $\quad H_i = H_i^{free} \oplus H_i^{tors}$

$$H^i = H^i_{free} \oplus H^i_{tors}$$

Then $\operatorname{rank} H_i^{free} = \operatorname{rank} H^i_{free}$

torsion ... ne $\quad \Big| \quad H_i^{tors} \cong H^{i+1}_{tors}$

torsion jumps!

$H_{i} \cong H_{\text{tors}}$

[Universal coefficient theorem in cohomology]

<u>Proof</u>: Alg. fact: any complex of free abelian groups splits as a direct sum of

$$0 \to \mathbb{Z} \to 0 \qquad\qquad 0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$$
$$(*) \qquad\qquad\qquad\qquad (*)$$

$H_i = \underset{\mathbb{Z}}{*}(*)$ in degree i $\oplus$

$\oplus\ \mathbb{Z}_m$ for all $0 \to \mathbb{Z}_{i+1} \xrightarrow{m} \mathbb{Z}_i \to 0$

Take duals:

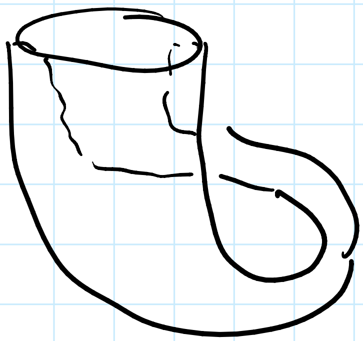
$$0 \leftarrow \mathbb{Z} \leftarrow 0 \qquad\qquad 0 \leftarrow \mathbb{Z}_{i+1} \xleftarrow{m} \mathbb{Z}_i \leftarrow 0$$

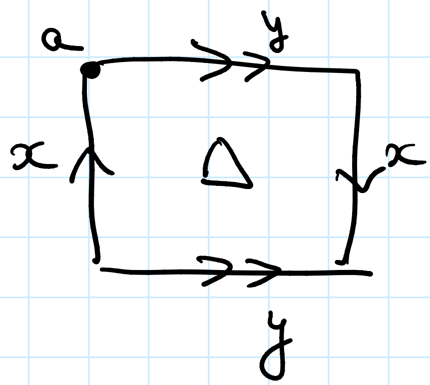
$\mathbb{Z}_m$ cohomology in degree $i+1$.

<u>Note</u>: If we take homology with coeffs in G, G = some abelian group, could be more complicated!

$G = \mathbb{Z}_p$, p prime (ex. $G = \mathbb{Z}_2$)
we work over a field $\mathbb{Z}_2 \Rightarrow$ OK.

Ex Klein bottle





Cell decomposition: one pt
two 1-cells x, y
one 2-cell Δ.

$$H_*:\quad \begin{array}{ccccccccc} 0 & \to & C_2 & \xrightarrow{\partial} & C_1 & \xrightarrow{\partial} & C_0 & \to & 0 \\ & & \parallel & & \parallel & & \parallel & & \\ & & \langle \Delta \rangle & & \langle x, y \rangle & & \langle a \rangle & & \end{array}$$

$$\partial(x) = 0 \quad \partial(y) = 0$$

$$\partial(\Delta) = x + y + x - y = 2x$$

$$H_0 = \mathbb{Z}\langle a \rangle \qquad H_1 = \underset{\langle y \rangle}{\mathbb{Z}} \oplus \underset{\langle x \rangle}{\mathbb{Z}_2} \qquad H_2 = 0$$

$$0 \to \mathbb{Z}_\Delta \xrightarrow{2} \begin{matrix} \mathbb{Z}_x \\ \oplus \end{matrix} \quad 0 \to \mathbb{Z}_a \to 0$$

$$0 \longrightarrow \mathbb{Z}_y \longrightarrow 0$$

H^*: $0 \leftarrow \underbrace{C^2}_{} \xleftarrow{\delta} C^1 \xleftarrow[0]{\delta} C^0 \leftarrow 0$

$H^0 = \mathbb{Z}$

$\varphi \in C^1$ $\quad \delta\varphi(\Delta) = \varphi(\partial\Delta) =$

$= \varphi(2x) = \underline{2\varphi(x)}$

$\delta\varphi = 0 \iff \varphi(x) = 0$

$\varphi(y) =$ anything $\quad H^1 = \mathbb{Z}$

Image of δ = all functions on Δ divisible by 2

$H^2 = \mathbb{Z}_2$

$$\boxed{\begin{matrix} H_0 = \mathbb{Z} & H_1 = \mathbb{Z} \oplus \mathbb{Z}_2 & H_2 = 0 \\ H^0 = \mathbb{Z} & H^1 = \mathbb{Z} & H^2 = \mathbb{Z}_2 \end{matrix}}$$

Klein bottle is non-orientable

Know $H_0(X) = \mathbb{Z}^{\#\text{connected components}}$

$H^0(X) = \mathbb{Z}^{\#\text{conn. components}}$

$H_1(X) = \dfrac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]} \rightsquigarrow$ can compute H^1

H^1 has no torsion (no torsion in H_0)

rank H^1 = rank of free part of H_1

Fact $H^1(X) = \mathrm{Hom}(\pi_1(X), \mathbb{Z})$

Torsion in H^2 = Torsion in H_1

Torsion in H^2 = Torsion in H_1

Functoriality $X \xrightarrow{f} Y$ continuous map

$$C_\bullet(X) \xrightarrow{f_*} C_\bullet(Y) \quad \text{chain map}$$

$$H_*(X) \xrightarrow{f_*} H_*(Y)$$

goes in the opposite way!

$$\left.\begin{array}{l} C^\bullet(X) \xleftarrow{f^*} C^\bullet(Y) \\ H^*(X) \xleftarrow{f^*} H^*(Y) \end{array}\right\} \text{dual map to } f_*$$

α = cocycle in $H^i(Y)$

x = cycle in $H_i(X)$

f_* = "pushforward"

f^* = "pullback"

$$f^*\alpha(x) = \alpha(f_* x)$$

Can write all long exact sequences (pair, Mayer-Vietoris...) in cohomology, work exactly in the same way, all maps go in the opposite direction.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$(g \circ f)_* = g_* \circ f_*$$

If $f, f' : X \to Y$ are homotopic

$(g \circ f)_* = g_* \circ f_*$

$(g \circ f)^* = f^* \circ g^*$

are homotopic then $f^* = (f')^*$

$f_* = f'_*$

in (co) homology.

$X \xrightarrow{\Delta} X \times X$

diagonal map $\quad p \longrightarrow (p, p)$

$H_*(X) \xrightarrow{\Delta_*} H_*(X \times X)$

$\underbrace{H^*(X \times X)} \xrightarrow{\Delta^*} H^*(X) \longleftarrow$ this will gives us cup product on cohomology!
