Lecture 21 (5/14)

Friday, May 14, 2021 2:06 PM

Differential forms.

① Differential forms on $\mathbb{R}^k$

$\Omega^\bullet(\mathbb{R}^n)$ = algebra generated by smooth functions on $\mathbb{R}^n$ $f(x_1, \ldots x_n)$ ← deg 0

and $\underbrace{dx_1, dx_2, \ldots dx_n}_{\text{degree } 1}$

Contains all possible products and linear combinations, we require sign rule for products

$$a \wedge b = (-1)^{\deg a \cdot \deg b} \, b \wedge a$$

⇒ functions commute with everything

⇒ dx_i anticommute

Product : $\alpha \wedge \beta$ ← notation for product.

$dx_i \wedge dx_i = 0$

$dx_i \wedge dx_j = -dx_j \wedge dx_i$

Ex $\underset{(x,y)}{\mathbb{R}^2}$: $\Omega^0 = \{f(x,y)\} = 0\text{-forms}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ || function (smooth)

$\Omega^1 = \{a(x,y)dx + b(x,y)dy\} = 1\text{-forms}$

$\Omega^2 = \{c(x,y)dx \wedge dy\}$ $\quad dy \wedge dx = -dx \wedge dy$

$\Omega^i = 0 \qquad \underline{i > 2}$

In general, $\Omega^i = 0 \quad i > n$

$\Omega^n(\mathbb{R}^n) = \{f(x_1 \ldots x_n) \cdot dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n\}$

Ex $\underset{\Omega^0}{f(x_1 \ldots x_n)} \rightsquigarrow df \in \Omega^1(\mathbb{R}^n)$ differential of f

$$\boxed{df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i}$$

② De Rham differential

$d : \Omega^k \longrightarrow \Omega^{k+1}$

Defined by the following properties:

- $\underset{\text{function}}{f \in \Omega^0} \longrightarrow \underset{\text{as above.}}{df \in \Omega^1}$

- $d(dx_i) = 0$
- Leibniz rule:

$$\boxed{d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d(\beta).}$$

Ex: $\omega = a(x,y)dx + b(x,y)dy \in \Omega^1(\mathbb{R}^2)$

$$d(\omega) = d(a \wedge dx) + d(b \wedge dy) = (\text{Leibniz})$$

$$= d(a) \wedge dx + (-1)^{\overset{0}{\deg a}} a \wedge \cancel{d(dx)} + d(b) \wedge dy +$$

$$+ (-1)^{\overset{0}{\deg b}} b \wedge \cancel{d(dy)} =$$

$a(x,y)dx$
$\overset{\shortparallel}{a(x,y) \wedge dx}$

$$= d(a) \wedge dx + d(b) \wedge dy =$$

$$= \left(\frac{\partial a}{\partial x} dx + \frac{\partial a}{\partial y} dy\right) \wedge dx + \left(\frac{\partial b}{\partial x} dx + \frac{\partial b}{\partial y} dy\right) \wedge dy$$

$$= \frac{\partial a}{\partial y} dy \wedge dx + \frac{\partial b}{\partial x} dx \wedge dy =$$

$$= \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy.$$

Lemma $d^2 = 0$, so we have a chain complex $\to \Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{d} \Omega^{k+2} \to \dots$

Proof: $\bullet\ d(d(f)) = ?$

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$$d(f) = \sum_i \frac{\partial f}{\partial x_i} \wedge dx_i$$

$$d(d(f)) = \sum_i d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i + \sum_i \frac{\partial f}{\partial x_i} \wedge \cancel{d(dx_i)\cdot(-1)^{\cdots}}$$

$$= \sum_i d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i = \sum_i \left(\sum_j \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j\right) \wedge dx_i$$

$i = j$ $\quad \frac{\partial^2 f}{\partial x_i^2} dx_i \wedge dx_i = 0$

$i \neq j$ $\quad \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \wedge dx_i + \frac{\partial^2 f}{\partial x_j \partial x_i} dx_i \wedge dx_j =$

$dx_i \wedge dx_j = -dx_j \wedge dx_i$

$$= \underbrace{\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right)}_{= 0} dx_i \wedge dx_j = 0$$

- $d^2(dx_i) = d(d(dx_i)) = 0$
- $d^2(\alpha \wedge \beta) = d(d(\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d(\beta))$

$$= d^2(\alpha) \wedge \beta + (-1)^{\deg d(\alpha)} d(\alpha) \wedge d(\beta) +$$

$$+ (-1)^{\deg \alpha} d(\alpha) \wedge d(\beta) + (-1)^{2\deg \alpha} \alpha \wedge d^2(\beta)$$

cancel out since

$\deg(d\alpha) = \deg(\alpha) + 1$

$\deg(d\alpha) = \deg(\alpha)+1$

$\Rightarrow$ if $d^2(\alpha) = d^2(\beta) = 0 \Rightarrow d^2(\alpha\wedge\beta) = 0$ ∎

③ Functoriality

$$\underset{x_1 \dots x_n}{\mathbb{R}^n} \xrightarrow{\varphi} \underset{y_1 \dots y_k}{\mathbb{R}^k}$$

$$\varphi^*: \Omega^\bullet(\mathbb{R}^k) \longrightarrow \Omega^\bullet(\mathbb{R}^n)$$

homomorphism, commutes with d.

$y_1 = \varphi_1(x_1, \dots x_n)$
$y_2 = \varphi_2(x_1, \dots x_n)$
$\vdots$
$y_k = \varphi_k(x_1, \dots x_n)$

• Given a function in y_i plug in $y_i = \varphi_i$

$$f \longrightarrow \underset{\varphi^* f}{\underset{\shortparallel}{f(\varphi(x_1, \dots x_n))}}$$

• $\varphi^*(dy_i) = d\varphi_i$

• Chain Rule $\Rightarrow d(\varphi^*(f)) = \varphi^*(df)$.

Ex If we change coordinates in $\mathbb{R}^n$ we know how the differential forms change.

Ex Polar coordinates: $x = r\cos\varphi$
$y = r\sin\varphi$

$$y = r\sin\varphi$$

$$\omega = a(x,y)\,dx + b(x,y)\,dy$$

How to write it in polar coords?

$$dx = \cos\varphi\, dr - r\sin\varphi\, d\varphi$$

$$dy = \sin\varphi\, dr + r\cos\varphi\, d\varphi$$

$$\omega = a(r\cos\varphi, r\sin\varphi)(\cos\varphi\, dr - r\sin\varphi\, d\varphi) +$$
$$+ \; b(r\cos\varphi, r\sin\varphi)(\sin\varphi\, dr + r\cos\varphi\, d\varphi).$$

Functoriality

$$\begin{tikzcd} \omega_{x,y} \arrow[r, "\text{change}", "\text{coords}"'] \arrow[d, "d"'] & \omega_{r,\varphi} \arrow[d, "d"] \\ d\omega_{x,y} \arrow[r, "\text{change}", "\text{coords}"'] & d\omega_{r,\varphi} \end{tikzcd}$$

④ Forms on manifolds

M = <u>smooth</u> n-manifold

$\Omega^k(M)$ = k-forms on M

= objects which look like

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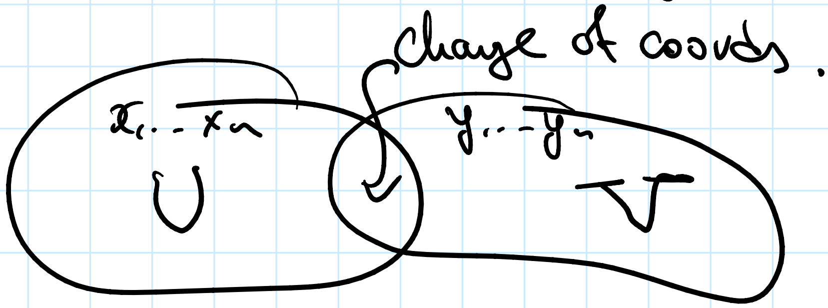
$\Omega^k(\mathbb{R}^n)$ in every coordinate chart.



$$\omega_{x_1 \ldots x_n} \longleftrightarrow \omega_{y_1 \ldots y_n}$$

on the intersection, related by change of coordinates $(x_1, \ldots x_n) \longleftrightarrow (y_1, \ldots y_n)$.

<u>Properties:</u> 1) $\Omega^\bullet(M)$ is a graded algebra

2) $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$

satisfying Leibniz rule and $d^2 = 0$

(can define in charts)

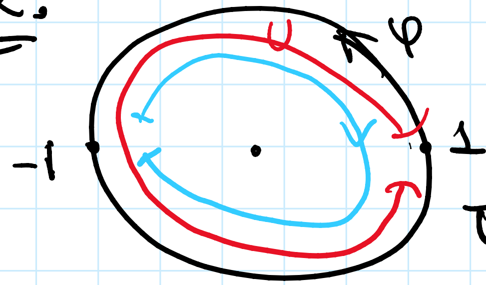
3) $M \xrightarrow{\varphi} N$

smooth map

$\varphi^* : \Omega^k(N) \longrightarrow \Omega^k(M)$

homomorphism, commutes with d.

Ex: S^1



$\varphi \sim \varphi + 2\pi$

local coordinate

$U = S^1 - \{1\}$

$V = S^1 - \{-1\}$

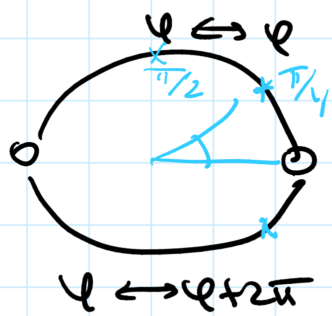
$-\pi/4 + 2\pi$

Coordinate in $U = \varphi \in [0, 2\pi]$

Coordinate in $V = \varphi \in [-\pi, \pi]$

$-\pi/4$

$U \cap V =$



$\Omega^\bullet(S^1)$: Ω^0 = functions on S^1

= periodic functions

$$\boxed{f(\varphi) = f(\varphi + 2\pi)}.$$

Ω^1:

$$\boxed{d\varphi = d(\varphi + 2\pi)}$$

$\Rightarrow d\varphi$ is a well defined form on S^1

(but φ is not a function!)

(but φ is not a function;)

Any one-form on S^1 is

$\boxed{\omega = f(\varphi)\, d\varphi}$ where $f(\varphi)$ is <u>periodic</u>.

change coord $\varphi \leftrightarrow \varphi + 2\pi$

$\omega = f(\varphi + 2\pi)\, d\varphi$

$\Omega^i = 0$ for $i > 1$.

<u>Rmk</u> If $f(\varphi) = f(\varphi + 2\pi)$

then $f'(\varphi) = f'(\varphi + 2\pi)$

and $df(\varphi) = f'(\varphi)\, d\varphi$.