Lecture 22 (5/17)

Monday, May 17, 2021 1:59 PM

Last time:

M = a smooth n-manifold

$\Omega^k(M)$ differential k-forms

$d: \Omega^k(M) \to \Omega^{k+1}(M)$

$d^2 = 0$

Def de Rham cohomology is the cohomology of $(\Omega^\bullet(M), d)$

$$H^k_{dR} = \frac{\mathrm{Ker}(d)}{\mathrm{Im}(d)}$$

both infinite dimensional

Ex $M = \mathbb{R}$

$$\begin{array}{l} \Omega^0(\mathbb{R}) = \{f(x)\} \\ \quad \downarrow d \\ \Omega^1(\mathbb{R}) = \{g(x)\,dx\} \end{array}$$

$d(f(x)) = f'(x)\,dx$.

$H^0_{dR} = \mathrm{Ker}\, d\,|_{\Omega^0} = \{f(x) \mid f'(x) = 0\}$

$= \{$ constant,

$= \left\{ \begin{array}{c} \text{constant} \\ \text{functions} \end{array} \right\} \simeq \mathbb{R}$

H^1_{dR}: $\underline{\mathrm{Ker}\, d = \Omega^1(\mathbb{R})}$

$\mathrm{Im}\, d$: $\{g(x)dx\}$ where $g = f'(x)$

By Fundamental Theorem of Calculus, $f(x) = \int_0^x g(t)\,dt$

$f'(x) = g(x)$

$\Rightarrow \underline{\mathrm{Im}\, d = \Omega^1(\mathbb{R})} \Rightarrow H^1_{dR}(\mathbb{R}) = 0$

Conclusion: $H^0_{dR}(\mathbb{R}) = \mathbb{R}$,

$H^i_{dR}(\mathbb{R}) = 0, \ i > 0$.

Ex $M = S^1 \quad \varphi \sim \varphi + 2\pi$

$$\Omega^0(S^1) = \{ f(\varphi) : f(\varphi) = f(\varphi + 2\pi) \}$$

$\big\downarrow d$ 2π-periodic functions on $\mathbb{R}$

$$\Omega^1(S^1) = \{ g(\varphi)d\varphi : g(\varphi) = g(\varphi + 2\pi) \}.$$

$H^0_{dR} = \mathrm{Ker}\, d = \left\{ f : \begin{array}{c} f' = 0 \\ 2\pi\text{-periodic} \end{array} \right\}$

$\Rightarrow \{ \text{constant} \}$

2π-periodic $\}$

$$\Rightarrow \{\text{constant functions}\} \cong \mathbb{R}$$

$$H^1_{dR} = \frac{\Omega^1(S^1)}{\mathrm{Im}\, d}. \qquad g(\varphi)\,d\varphi \overset{?}{=} d(f(\varphi))$$

$$g(\varphi) \overset{?}{=} f'(\varphi).$$

$$f(\varphi) = \int_0^{\varphi} g(t)\,dt$$ by fund. thm of Calculus.

If $g(\varphi)$ is 2π-periodic, $f(\varphi)$ might be <u>not</u> periodic.

In fact, f is 2π-periodic

$$\Updownarrow$$

$$\int_0^{2\pi} g(t)\,dt = 0$$

$$\mathrm{Im}\, d \longrightarrow \Omega^1(S^1) \xrightarrow{\ I\ } \mathbb{R}$$

$$I(g(\varphi)\,d\varphi) = \int_{S^1} g(\varphi)\,d\varphi = \int_0^{2\pi} g(\varphi)\,d\varphi.$$

$$H^1_{dR}(S^1) = \frac{\Omega^1(S^1)}{\mathrm{Im}\, d} = \frac{\Omega^1(S^1)}{\mathrm{Ker}\, I} = \mathbb{R}.$$

$\overline{\mathrm{Im}\, d} = \overline{\mathrm{Ker}\, I} = \mathbb{R}$.

For example $\int_{S^1} d\varphi = \int_0^{2\pi} d\varphi = 2\pi \neq 0$

$\Rightarrow \left[\frac{1}{2\pi} d\varphi\right]$ is a generator in $H^1_{dR}(S^1)$.

Conclusion: $H^0_{dR}(S^1) = H^1_{dR}(S^1) = \mathbb{R}$

$H^i_{dR}(S^1) = 0 \quad , i > 0.$

Thus (de Rham) For any compact n-manifold M, $\boxed{H^i_{dR}(M) = H^i(M; \mathbb{R})}$

Ideas in the proof:

① Poincaré lemma: $H^i_{dR}(\mathbb{R}^n) = \begin{cases} 0 & , i > 0 \\ \mathbb{R} & , i = 0. \end{cases}$

$\alpha \in \Omega^k(\mathbb{R}^n) \qquad k > 0$

if $d\alpha = 0$ then $\alpha = d\beta$
(for some $\beta \in \Omega^{k-1}$).

$k = 0$ easy: $\alpha = f(x_1 \ldots x_n)$

$d\alpha = \sum \frac{\partial f}{\partial x_i} dx_i$

$$d\alpha = \sum \frac{\partial f}{\partial x_i} dx_i$$

$$d\alpha = 0 \Leftrightarrow \frac{\partial f}{\partial x_i} = 0 \quad \text{for all } i$$

$$\Leftrightarrow f = \text{const}.$$

<u>$k=1$</u>: $\alpha = \sum_i f_i \, dx_i$ $\quad f_i$ = some function

$$d\alpha = \sum_i df_i \wedge dx_i =$$

$$= \sum_i \Big(\sum_j \frac{\partial f_i}{\partial x_j} dx_j \Big) \wedge dx_i$$

$$= \sum_{i<j} \Big(\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} \Big) dx_j \wedge dx_i$$

$dx_i \wedge dx_j =$
$= -dx_j \wedge dx_i$

Poincaré lemma in this case:

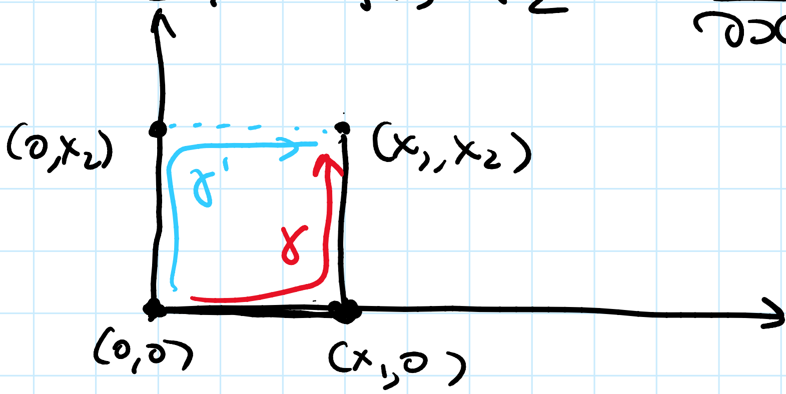
If $\boxed{\dfrac{\partial f_i}{\partial x_j} = \dfrac{\partial f_j}{\partial x_i}}$ for all $i \neq j$

then $\exists\, h$ such that $\dfrac{\partial h}{\partial x_i} = f_i$

<u>How to find h?</u> Let us do it

for $n=2$. $\quad f_1, f_2 \quad \dfrac{\partial f_1}{\partial x_2} = \dfrac{\partial f_2}{\partial x_1}$

for $n=2$. f_1, f_2 $\quad \dfrac{\partial f_1}{\partial x_2} = \dfrac{\partial f_2}{\partial x_1}$



γ = path $(0,0) \to (x_1,0) \to (x_1,x_2)$

$$\alpha = f_1\,dx_1 + f_2\,dx_2$$

$$h = \int_\gamma \alpha = \int_0^{x_1} f_1(t,0)\,dt + \int_0^{x_2} f_2(x_1,t)\,dt$$

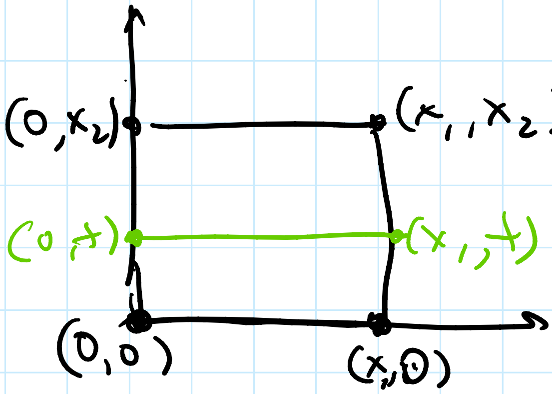
Clearly, $\dfrac{\partial h}{\partial x_2} = f_2(x_1,x_2)$ by fund. thm of Calculus.

__Claim__ $$\int_\gamma \alpha = \int_{\gamma'} \alpha = \int_0^{x_2} f_2(0,t)\,dt + \int_0^{x_1} f_1(t,x_2)\,dt.$$

__Proof__ $$\boxed{\int_\gamma \alpha - \int_{\gamma'} \alpha = \int_{\blacksquare} d\alpha}$$

Stokes formula.

What does it mean?



$$d\alpha = \frac{\partial f_1}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial f_2}{\partial x_1} dx_1 \wedge dx_2$$

$$= \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2$$

$$\int_{▨} d\alpha = \int_0^{x_1}\int_0^{x_2} \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 dx_2$$

$$\int_0^{x_1}\int_0^{x_2} \frac{\partial f_2}{\partial x_1} dx_1 dx_2 = \int_0^{x_2}\int_0^{x_1} \frac{\partial f_2}{\partial x_1} dx_1 dx_2$$

$$= \int_0^{x_2} \left[f_2(0,t) - f_2(x_1,t) \right] dt$$

The same for $\frac{\partial f_1}{\partial x_2}$, so if we add this up we get

$$\iint_{▨} d\alpha = \int_\gamma \alpha - \int_{\gamma'} \alpha = 0$$

since $d\alpha = 0$.

$$h = \int \alpha = \int \alpha'$$

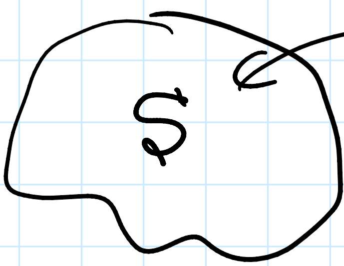
$$h = \int_{\gamma} \alpha = \int_{\gamma'} \alpha'$$

$\frac{\partial h}{\partial x_1} = f_1$

satisfies $\frac{\partial h}{\partial x_2} = f_2$

Assuming $d\alpha = 0$, we found h such that $dh = \alpha$.

In general, the proof of Poincaré lemma is similar.



subset in $\mathbb{R}^k$ (nice enough, open)

$$\alpha \in \Omega^k(\mathbb{R}^k) = f(x_1 \ldots x_k)\, dx_1 \wedge \ldots \wedge dx_k$$

$$\Rightarrow \int_S \alpha = \int_S f(x_1 \ldots x_k)\, \underline{dx_1 \ldots dx_k}$$

usual multivariable integral.

<u>Rmk</u> Change of variables in multivariable integral $\Longleftrightarrow$ this does not depend on the choice of coords!

coords.

Recall: MV calculus: $f \rightsquigarrow f \cdot \text{Jac}$ if we change variables

The Jacobian naturally appears when we deal with diff. forms:

$$\begin{matrix} x_1 & & y_1(x_1 \dots x_k) \\ \vdots & \longleftrightarrow & \vdots \\ x_k & & y_k(x_1 \dots x_k) \end{matrix}$$

$$dy_1 = \frac{\partial y_1}{\partial x_1} dx_1 + \dots + \frac{\partial y_1}{\partial x_k} dx_k$$

$$\vdots$$

$$dy_k = \frac{\partial y_k}{\partial x_1} dx_1 + \dots + \frac{\partial y_k}{\partial x_k} dx_k.$$

Exercise $dy_1 \wedge \dots \wedge dy_k =$

$$= \det\left(\frac{\partial y_i}{\partial x_j}\right) \cdot dx_1 \wedge \dots \wedge dx_k.$$

$\mathbb{R}^n$, $\phi : \mathbb{R}^k \longrightarrow \mathbb{R}^n$

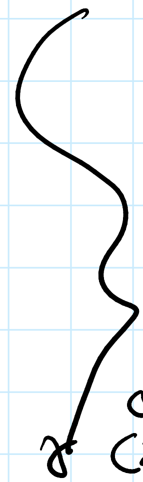
α = k-form on $\mathbb{R}^n$

$\phi^*\alpha$ = k-form on $\mathbb{R}^k$

$\phi^*\alpha$ = k-form on $\mathbb{R}^k$

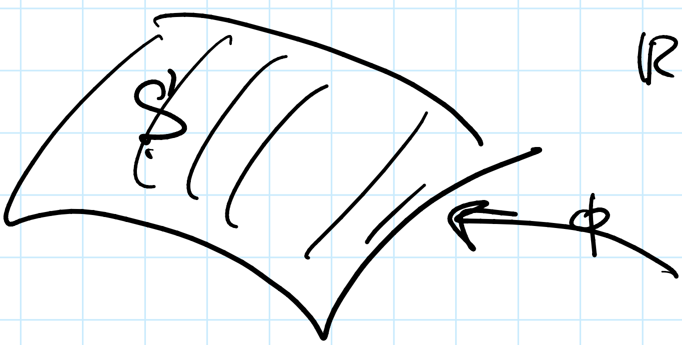
S = some subset of $\mathbb{R}^k$

$$\int_S \phi^*\alpha =: \int_{\phi(S)} \alpha$$



$$\int_\gamma \alpha = \int_{\mathbb{R}^1} \gamma^*\alpha = \int a(x(t),y(t))\,dx(t) + \int b(x(t),y(t))\,dy(t)$$

$$= \int \left[a(x(t),y(t))\,x'(t) + b(x(t),y(t))\,y'(t)\right]dt,$$



$\mathbb{R}^3$

$\alpha \in \Omega^2(\mathbb{R}^3)$

$\phi^*\alpha \in \Omega^2(\mathbb{R}^2)$

$$\int_S \alpha = \int_{\mathbb{R}^2} \phi^*\alpha$$