Lecture 23 (5/19)

Wednesday, May 19, 2021 2:09 PM

We can integrate differential forms:

(1) $S \subset \mathbb{R}^k$ $\quad \alpha = k$-form on S

$$\alpha = f(x_1 \ldots x_k)\, dx_1 \wedge \ldots \wedge dx_k$$

in <u>some</u> coordinates

$$\int_S \alpha = \int_S f(x_1 \ldots x_k)\, dx_1 \ldots dx_k .$$

<u>Fact</u> This does not depend on the choice of coordinates.

(2) $\underset{\mathbb{R}^k}{\overset{S}{\cap}} \xrightarrow{\varphi} M^n$ $\quad \alpha = k$-form on M

$\varphi^* \alpha = k$-form on S

$\int_S \varphi^* \alpha$ is well defined.

<u>Ex</u> $\sigma : \Delta^k \longrightarrow M^n$ (smooth) singular simplex

simplex

If α is a k-form on M, then

$\int_\Delta \sigma^* \alpha$ is a well defined real number

(depends on orientation of simplex).

$\rightsquigarrow$ α defines a function on singular chains!

$$\sum c_i \sigma_i \longrightarrow \sum c_i \int_\Delta \sigma_i^* \alpha$$

singular k-simplices $\Rightarrow$ $\alpha \longleftrightarrow C^k(M)$

singular cochains = functions on singular chains.

Stokes Theorem: $\sigma : \Delta^{k+1} \longrightarrow M$

$$\int_\Delta \sigma^*(d\alpha) = \int_{\partial\Delta} \sigma^* \alpha$$

$$\|$$

$$\int_\Delta d(\sigma^* \alpha)$$

α is a k-form

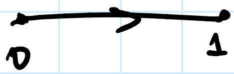
$d\alpha$ is a $k+1$ form

$\partial\Delta$ is a union of k-simplices (w. signs)

This is a multidimensional version of the

$\boxed{\Delta}$ This is a multidimensional version of the fundamental theorem of Calculus.

Ex $\Delta = [0,1]$



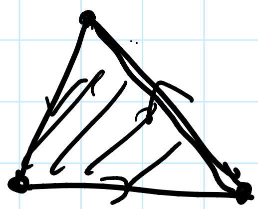
$$\alpha = f(x) \in \Omega^0(\mathbb{R})$$

$$d\alpha = f'(x)\,dx \in \Omega^1(\mathbb{R})$$

$$\int_\Delta d\alpha = \int_0^1 \underbrace{f'(x)\,dx}_{\text{1-form } d\alpha} \overset{}{\underset{\text{Stokes}}{=}} \int_{\partial\Delta} \alpha = \int_{\{1\}} f(x) - \int_{\{0\}} f(x)$$

$$= f(1) - f(0)$$

Ex



$\Delta = 2$-simplex in $\mathbb{R}^2$

$$\alpha = a(x,y)\,dx + b(x,y)\,dy$$

$$d\alpha = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy$$

$$\int_\Delta d\alpha = \int_\Delta \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx\,dy = (\text{Stokes})$$

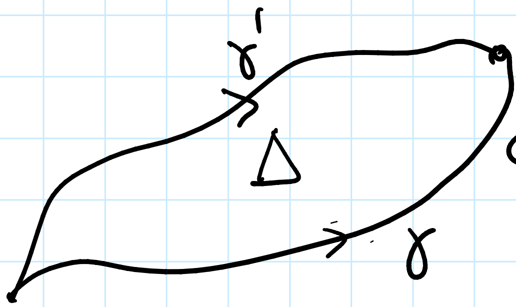
$$\| $$

$$\int_{\partial\Delta} \alpha = \int a(x(t),y(t))\,x'(t)\,dt + \int b(x(t),y(t))\,y'(t)\,dt$$

parametrize $\partial\Delta$ by $\gamma(t)$

$$\boxed{\int_\Delta d\alpha = \int_{\partial\Delta} \alpha}$$

Ex If $\boxed{d\alpha = 0}$ then $\int_{\partial\Delta} \alpha = 0$



$\Leftrightarrow$ $\int_\gamma \alpha = \int_{\gamma'} \alpha$

provided that γ, γ' are in the same homology class.

$$\partial\Delta = \gamma - \gamma'$$

$$\int_\gamma \alpha - \int_{\gamma'} \alpha = \int_{\partial\Delta} \alpha = (\text{Stokes}) = \int_\Delta d\alpha = 0$$

α is k-form on M $\Rightarrow$ cochain in $C^k(M)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ||
function on singular k-chains.

$\int_\Delta \alpha$

Stokes:

$$\int_\Delta d\alpha = \int_{\partial\Delta} \alpha \Rightarrow$$ de Rham differential d is dual to the differential on $C_*(M)$

the differential
on $C_k(M)$

<u>Thm</u> The above construction gives a chain map:

$$(\Omega^\bullet(M), d) \longrightarrow (C^k(M), \delta)$$

singular cochains

This gives a map in cohomology:

$$\underset{H^*_{dR}(M)}{\underset{\parallel}{H^*(\Omega^\bullet(M), d)}} \longrightarrow \underset{H^*(M; \mathbb{R})}{\underset{\parallel}{H^*(C^k(M), \delta)}}$$

singular cohomology.

<u>Thm</u> (de Rham) This map on cohomology

$$H^*_{dR}(M) \longrightarrow H^*(M; \mathbb{R})$$

is an isomorphism.

<u>Cor:</u> $H^\bullet_{dR}(M)$ is finite-dim and can be computed explicitly.

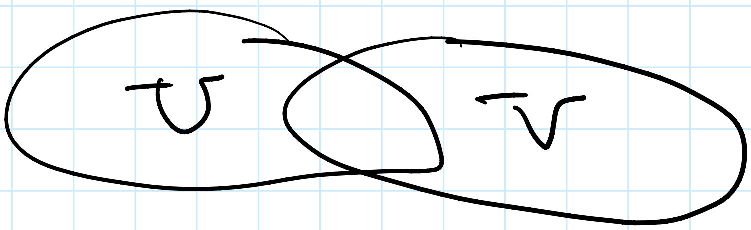
<u>Idea of proof:</u> ① Prove for $\mathbb{R}^n$, convex subsets in $\mathbb{R}^n$ (Poincaré lemma)

subsets in $\mathbb{R}^n$ (Poincaré lemma)

contractible, so $H^*(S, \mathbb{R}) = H^*(pt)$
need to prove same for H^*_{dR}.

② Develop Mayer-Vietoris sequence

$M = U \cup V$



$$0 \to \Omega^k(M) \longrightarrow \Omega^k(U) \oplus \Omega^k(V) \longrightarrow \Omega^k(U \cap V) \to 0$$

maps = restrictions of forms to subsets.

A form on M = pair (form on U, form on V) which agree on the intersection.

③ Compare this to Mayer-Vietoris sequence in cohomology, use 5-lemma. ∎

Facts and consequences:

① M^n is orientable if and only if there is a nonvanishing n-form $\omega \in \Omega^n(M^n)$

$\omega \neq 0$ at every point of M.

② If M is orientable (compact, no boundary)

$\Rightarrow$ can define $\int_M \omega$ for any n-form ω.

$= \int_{[M]} \omega \qquad [M] \in H_n(M)$

$$\Omega^n(M) \longrightarrow \mathbb{R}$$
$$\omega \longrightarrow \int_M \omega$$

We know that $H^n(M, \mathbb{R}) = \mathbb{R}$

$\Rightarrow$ for any ω such that $\int_M \omega = 0$

we have $\omega = d\alpha$ for some $\alpha \in \Omega^{n-1}(M)$

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<u>Note:</u> If $\omega \neq 0$ at every point of M then we can choose local coordinates such that $\omega = f(x_1 \dots x_n) dx_1 \wedge \dots dx_n$ and $f > 0 \Rightarrow \int f(x_1 \dots x_n) dx_1 \dots dx_n > 0$

$$\Rightarrow \int_M \omega > 0$$

<u>Cor</u> Any nowhere vanishing n form on M^n represents a nontrivial cohomology class in $H^n(M, \mathbb{R}) = \mathbb{R}$

<u>Rmk</u> Assume that ω is a nowhere vanishing n-form on M. We say that a local coordinate system $(x_1 \dots x_n)$ is <u>positively</u> oriented if $\omega = f(x_1 \dots x_n) dx_1 \wedge \dots \wedge dx_n$ for $f > 0$.

One can check that this gives a well defined orientation on M
$\Rightarrow M$ is orientable.
(Sometimes ω is called a <u>volume form</u>).

(3) If M^n is a compact orientable manifold with boundary then

$$\int_M d\alpha = \int_{\partial M} \alpha \quad \text{for an } (n-1) \text{ form } \alpha$$

$$\int_M d\alpha = \int_{[M]} d\alpha$$

$[M] \leftarrow$ n-chain $\quad \partial [M] = [\partial M]$.

This is a generalization of Stokes theorem.