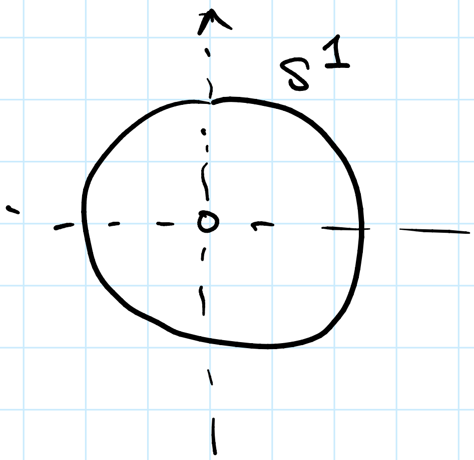
Lecture 24 (5/21)

Friday, May 21, 2021 1:53 PM

Ex $\mathbb{R}^2 \setminus \{(0,0)\} \sim S^1$



$H_0(\mathbb{R}^2 - 0) = H_1(\mathbb{R}^2 - 0) = \mathbb{Z}$

$\Rightarrow H^0(\mathbb{R}^2 - 0) = H^1(\mathbb{R}^2 - 0) = \mathbb{Z}$

generator of $H_1(\mathbb{R}^2 - 0) = [S^1]$

unit circle

What does it mean for de Rham cohomology?

$$\left.\begin{aligned} H^0_{dR}(\mathbb{R}^2 - 0) &= H^0(\mathbb{R}^2 - 0, \mathbb{R}) = \mathbb{R} \\ H^1_{dR}(\mathbb{R}^2 - 0) &= H^1(\mathbb{R}^2 - 0; \mathbb{R}) = \mathbb{R} \end{aligned}\right] \text{ by de Rham thm.}$$

What is the generator of H^1_{dR}?

$\alpha \in \Omega^1$, $\underline{d\alpha = 0}$

$$\int_{S^1} \alpha \neq 0$$

pairing between H^1_{dR} and H_1.

How to find such form?

How to find such form?

- Use polar coordinates (r, φ)

$$\alpha = d\varphi = d \arctan(y/x)$$

well defined locally, need to check that defines a form on $\mathbb{R}^2 - \{0\}$

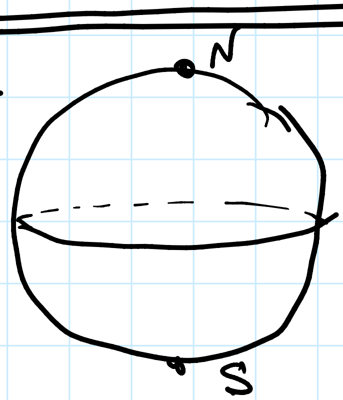
HW: explain why this is well defined everywhere on $\mathbb{R}^2 - \{0\}$

Check $d\alpha = 0$, compute $\int_{S^1} \alpha = \int_{S^1} d\varphi$

$\varphi \sim \varphi + 2\pi \quad d\varphi = d(\varphi + 2\pi)$

- Another method: use complex analysis and Cauchy residue formula.

Ex S^2



Two charts:

$U = S^2 - \{S\} \simeq \mathbb{R}^2$

$V = S^2 - \{N\} \simeq \mathbb{R}^2$

$U \cap V = \mathbb{R}^2 - 0$ which we just discussed.

By de Rham theorem, $H^2_{dR}(S^2) = H^2(S^2; \mathbb{R})$

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$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; \mathbb{R}$

Want to explain the mechanics

(A Mayer-Vietoris sequence in this case.

$\omega \in \Omega^2(S^2)$ automatically $d\omega = 0$

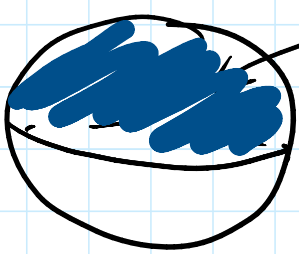
3-form on S^2

On U, we have $H^2_{dR} = 0$

$\Rightarrow$ can find a 1-form α_U such that

$d\alpha_U = \omega$ on U $\qquad$ defined on U

By Stokes theorem,



$$\int_{\text{North hemisphere}} \omega = \int_{N.h.} d\alpha_U =$$

$\partial(N.h) = S^1 =$ equator

$$(\text{Stokes}) = \int_{S^1} \alpha_U$$

Similarly, on V we can find another form α_V such that

$$d\alpha_V = \omega \text{ and } \int_{\text{South}} \omega = -\int_{S^1} \alpha_V$$

$$d\alpha_V = \omega \quad \text{and} \quad \int_{\substack{\text{South}\\ \text{hemisphere}}} \omega = -\int_{S^1} \alpha_V$$

$$\int_{S^2} \omega = \int_{N.h} \omega + \int_{S.h} \omega = \int_{S^1} (\alpha_U - \alpha_V)$$

pairing of $[\omega] \in H^2$ with $[S^2] \in H_2$

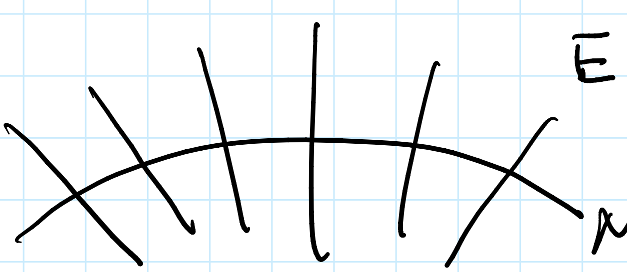
pairing of $[\alpha_U - \alpha_V] \in H^1_{dR}(U \cap V)$ with $[S^1] \in H_1(U \cap V)$

<u>Note</u>: $d(\alpha_U - \alpha_V) = d\alpha_U - d\alpha_V = \omega - \omega = 0$ on $U \cap V$!

<u>Cor</u>: (also from Mayer-Vietoris)

$$H^2_{dR}(S^2) \simeq H^1_{dR}(U \cap V) = H^1_{dR}(\mathbb{R}^2 \setminus \{0\})$$

Vector bundles



M = any top. space

Rank n vector bundle on M

$\approx$ family of vector spaces continuously depending on a point in M

- $E \xrightarrow{\pi} M$ total space

$\pi^{-1}(p) \cong \mathbb{R}^n$ for every point on M

- This $\mathbb{R}^n$ is a vector space:

$$\begin{tikzcd}
\mathbb{R}\times E \arrow[r, "\text{continuous}"] \arrow[d] & E \arrow[d] \\
\mathbb{R}\times M \arrow[r] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
(\underset{\text{number}}{\lambda}, v) \arrow[r, mapsto] \arrow[d] & \lambda\cdot v \arrow[d] \\
(\lambda, p) \arrow[r, mapsto] & p
\end{tikzcd}$$

v = vector in the fiber over p

λv = vector in the same fiber

v_1, v_2 = vectors in the fiber over p

$\Rightarrow v_1 + v_2$ = another vector in this fiber.

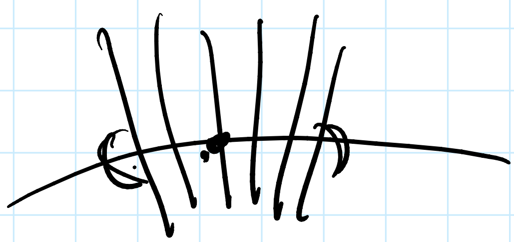
$$\begin{tikzcd}
\tilde{E} = \{(v_1, v_2) : \pi(v_1) = \pi(v_2)\} \arrow[d, "\text{continuous}"'] \\
E
\end{tikzcd}
\qquad
\begin{tikzcd}
(v_1, v_2) \arrow[d, mapsto] \\
v_1 + v_2
\end{tikzcd}$$

commutes with projection.

• Locally trivial condition:

for any point $p \in M$, there is a neighborhood U such that

$\pi^{-1}(U) = U \times \mathbb{R}^n$



Rmk If M is a smooth manifold, we can require that E is smooth and all structure maps are smooth.

Rmk We can consider vector bundles over different fields, for example, over $\mathbb{C}$ = complex vector bundle.

Ex $M \times \mathbb{R}^n$ is trivial bundle.

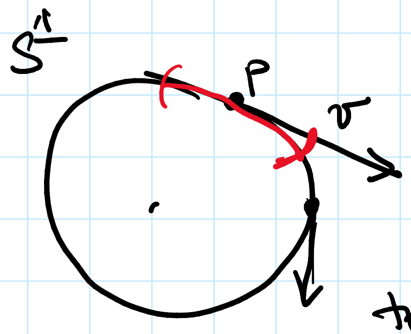
$$\begin{gathered} M \times \mathbb{R}^n \\ \| \\ (p, v) \quad p \in M,\ v \in \mathbb{R}^n \end{gathered}$$

$(p, v) \qquad p \in M, \; v \in \mathbb{R}^n$

$(p, v_1) + (p, v_2) = (p, v_1 + v_2)$.

<u>Ex</u> Tangent bundle to S^1

$TS^1 = \{(p, v)\}$



$p \in S^1$

v = tangent vector to S^1 at a point p.

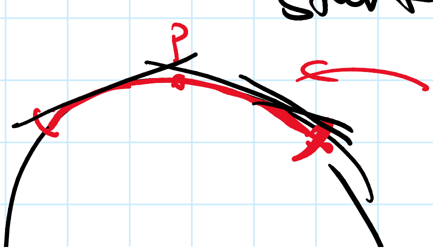
$$\begin{array}{ccc} TS^1 \subset S^1 \times \mathbb{R}^2 & & \\ \quad\; \downarrow \pi \qquad\; \Downarrow \quad\; \Downarrow & & \\ \quad\; S^1 \qquad\;\;\; p \quad\;\; v & & \end{array}$$

$(p, v) \mapsto p$

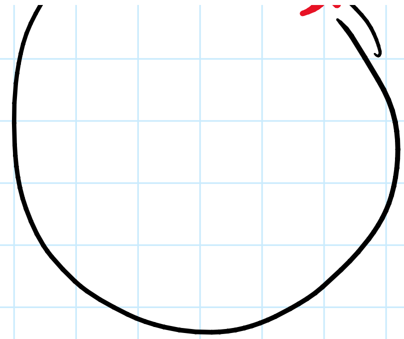
$\pi^{-1}(p) = \{$all tangent vectors at $p\}$
$\quad = $ tangent line at p

Can • multiply a vector by a number

• add two vectors if they start at the same point p



in this neighborhood, all tangent lines



all tangent lines project isomorphically to the horizontal line.

$$\pi^{-1}(U) \simeq \{(p \in S^1, v \in \text{horiz line}\}$$

$$= U \times \mathbb{R}$$

__Ex__: (next time) $TS^2 = \{p,(v)\}$

$p \in S^2$, v = vector in $\mathbb{R}^3$ tangent to S^2 at p.