Lecture 26 (5/26)

Wednesday, May 26, 2021 2:10 PM

$\pi : V \longrightarrow M$ rank r vector bundle

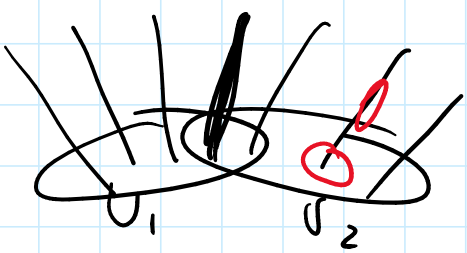
V ← total space

<u>Today</u> M is a smooth n-dimensional manifold

<u>Def</u> V is orientable if we can choose compatible orientation on all fibers of V ($\Longleftarrow$ $\pi^{-1}(U) = U \times \mathbb{R}^r$ for some nbhd U)

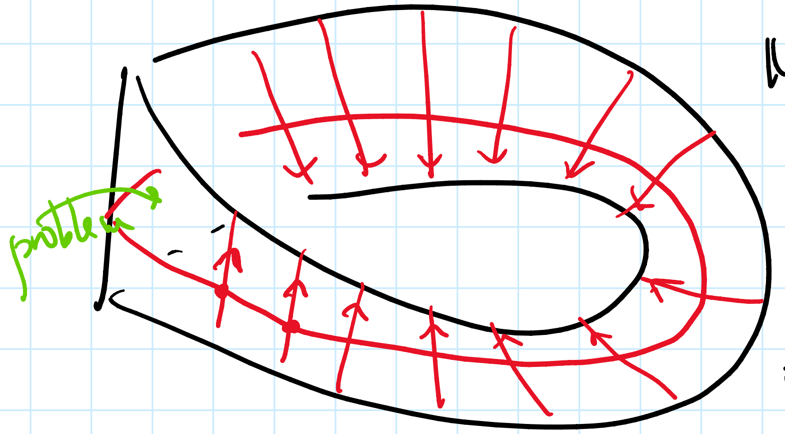
choose orientation on $\mathbb{R}^n$

Orient $\mathbb{R}^n$, so that it is compatible on $\pi^{-1}(U_1 \cap U_2)$



<u>Facts</u>: 1) If M is orientable & bundle is orientable then the total space of V is orientable. ($n+r$)-dim manifold.

In principle, can consider orientable bundles over non-orientable M.



Möbius band
$\|$
rank 1
bundle
over S^1
is **not** orientable

Ex Trivial bundle $M \times \mathbb{R}^n$
is *always* orientable for any M.

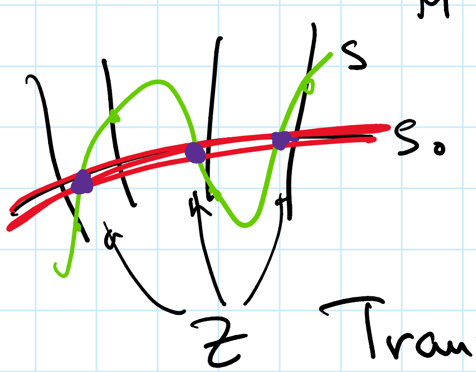
Euler class Construction:

① $s_0 : M \longrightarrow V$ zero section
$p \longrightarrow (p, 0)$

$s : M \longrightarrow V$ generic section
$p \longrightarrow (p, s(p))$

② Define

$$Z = s_0(M) \cap s(M) =$$
$$\overset{\shortparallel}{M}$$



$= \{ p : s(p) = 0 \}$

$=$ zero locus for section s

Transversality:

Z is a smooth submfld of M

Z is a smooth submanifold of M of dimension $\underbrace{n}_{\dim M} - \underbrace{r}_{\text{rank of bdle}}$ (if s is chosen in general position)

Note: locally $U \times \mathbb{R}^r$

$$s : U \longrightarrow \mathbb{R}^r$$

$\{s = 0\}$ = r equations on U

$\rightsquigarrow$ codim r submanifold if the equations are in general position.

Fact If the bundle V is oriented then Z is also oriented $\Rightarrow$ can define $[Z]$ = fundamental class of Z

$$[Z] \in H_{n-r}(Z) \longrightarrow H_{n-r}(M)$$

Def The Euler class of V is the class $e(V) \in H^r(M)$ Poincaré dual to $[Z]$.

[Assume M is oriented and bundle oriented]

oriented $\Rightarrow V$ oriented)

<u>Fact</u> $[Z]$ and $e(V)$ do not depend on the choice of generic section.
(submanifold Z does depend on s but its homology class does not).

<u>Ex</u> $r > n$ then $Z = \emptyset$
and $H^r(M) = 0$ and $e(V) = 0$

<u>Ex</u> $r = n$ then $\dim Z = 0$
$\Rightarrow Z$ is a bunch of points
with signs, then $e(V) \in H^r(M) = H_0(M) = \mathbb{Z}$
counts these intersection points with signs
$$e(V) = s_0(M) \circ s(M)$$
$\leftarrow$ intersection index.

That's what we discussed last time.

<u>Facts:</u> ① If V is not orientable
then can still define $e(V) \in H^r(M; \mathbb{Z}_2)$
(same construction, ignore orientations)

(same construction, ignore orientations)

② If V has a nowhere zero section then $e(V) = 0$

Proof: Choose this section as s, then $Z = \emptyset$

(if $e(V) \neq 0$ then V does not have a nonzero section, so $e(V)$ is an obstruction to existence of such section)

③ In particular, if V is trivial then $e(V) = 0$.

④ $V = V_1 \oplus V_2$ (fibers are direct sums of fibers)

$$r = r_1 + r_2$$

Then $e(V) = e(V_1) \cup e(V_2)$

Proof: Choose a section $s = (s_1, s_2)$

s_1 = section of V_1 $\quad Z_1 = \{s_1 = 0\}$
s_2 = section of V_2 $\quad Z_2 = \{s_2 = 0\}$

$$Z = \{s = 0\} = Z_1 \cap Z_2$$

$$PD[Z] = PD[Z_1] \cup PD[Z_2]$$
$$\overset{\parallel}{e}(V) = e(V_1) \cup e(V_2).$$

3) $V = V_1 \oplus \mathbb{R}^k$ ← trivial bdle

$$e(V) = e(V_1) \cup e(\mathbb{R}^k)$$
$$= e(V_1) \cup 0 = 0$$

Indeed, $\mathbb{R}^k$ has a nowhere zero section $s_2 \Rightarrow V$ has a section $(0, s_2)$

Thm The Euler class of TM equals the Euler characteristic of M.

Here TM is the tangent bundle, so $r = n$ and $e(TM) \in \mathbb{Z}$.

Proof 1) A section s of TM is a vector field on M (at every point of M we choose a tangent vector).

$Z = \{s = 0\}$ = set of zeroes of

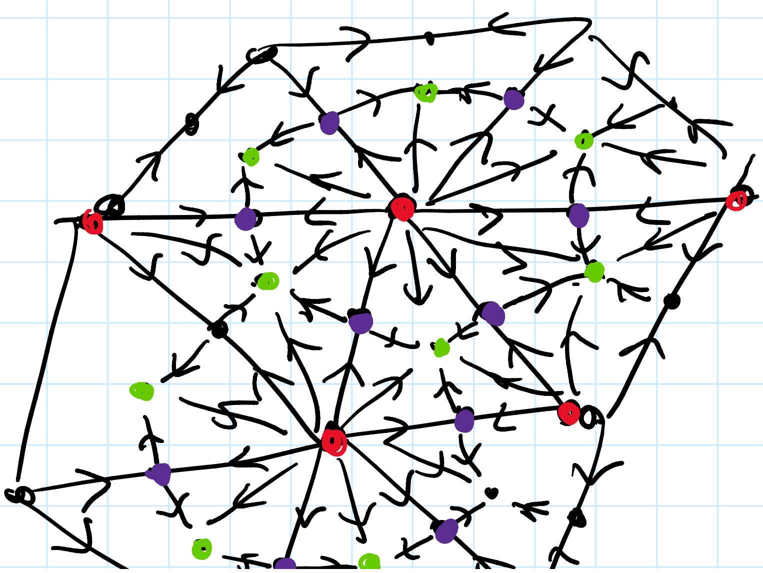
$Z = \{s = 0\}$ = set of zeroes of this vector field

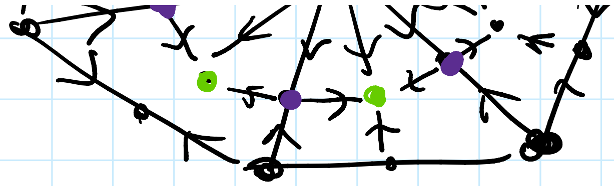
$[Z]$ = number of such zeroes (w. signs) with signs.

By the above, this does not depend on a choice of vector field (as long as it is generic).

② To prove the theorem, we need to choose one particular vector field and count zeros.

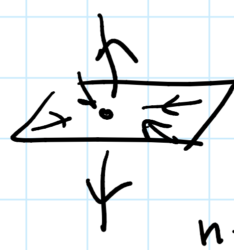
Assume $n = 2$, so M is a surface, triangulate:





Have a source at every vertex of triangulation, a saddle at every edge, and a sink at the barycenter of each triangle.

at a zero point:



k attracting directions, n−k repelling directions.

$$\mathrm{index}(\text{source}) = +1 = (-1)^0$$

$$\mathrm{index}(\text{saddle}) = -1 = (-1)^1$$

$$\mathrm{index}(\text{sink}) = +1 = (-1)^2$$

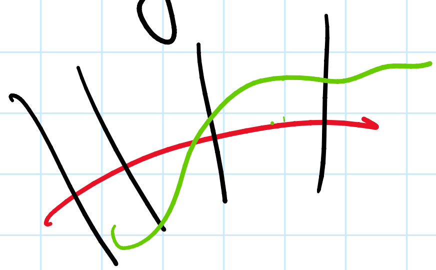
$$e(TM) = \sum_{\text{vert}} (+1) + \sum_{\text{edges}} (-1) + \sum_{\text{faces}} (+1) \qquad \mathrm{index} = (-1)^k$$

$$= V - E + F = \chi(M).$$

In general, do the same for any simplex of any dimension in a triangulation of M in a triangulation of M

Index (zero of s) $= (-1)^{\dim(\text{simplex})}$. ∎

Cor: If M has a nowhere vanishing vector field then $\chi(M)=0$.



orientation on M
$\Rightarrow$ orientation on $s(M)$
and on $s_0(M)$

Change orientation on M
$\Rightarrow$ change orientation on $s(M)$
$s_0(M)$
total space V

$\Rightarrow$ change orientation on Z

$$[Z] \longrightarrow -[Z]$$

$\Rightarrow$ change sign in Poincaré dual

$$e(V) \longleftrightarrow e(V).$$

Conclusion: Change of orientation on M preserves $e(V)$

- But: change of orientation on

- <u>But</u>: change of orientation on fibers changes $e(V) \longleftrightarrow -e(V)$