Lecture 27 (5/28)

Friday, May 28, 2021 2:10 PM

Euler class $\pi : V \longrightarrow M$

rank r vector bundle,

M = (compact) top. space,

not necessary a manifold.

Want to construct Euler class

$$e(V) \in H^r(M).$$

Idea: Orientation of a vector space $\mathbb{R}^n$ = class in $H_n(\mathbb{R}^n; \mathbb{R}^n - 0)$

$$H_n(\mathbb{R}^n; \mathbb{R}^n - 0) = H_{n-1}(\mathbb{R}^n - 0) = H_{n-1}(S^{n-1}) = \mathbb{Z}$$

We want to run this construction in every fiber of V:

Thm (Thom isomorphism) If V is an orientable vector bundle then

$$H^i(M) \simeq H^{i+r}(\bar{V}; \bar{V} \setminus M)$$

rank of bundle

total space minus zero section

If non-orientable, works with $\mathbb{Z}_2$ coefs.

<u>Idea of proof</u>: Trivialize V on small open subsets of M, choose finite cover of these.

$$\pi^{-1}(U) = U \times \mathbb{R}^r$$

$$H^*(U \times (\mathbb{R}^r - \{0\})) =$$

$$= (\text{Künneth}) = H^*(U \times S^{r-1})$$

$$= H^i(U) \otimes H^0(S^{r-1}) \oplus H^i(U) \otimes H^{r-1}(S^{r-1})$$

<u>So</u> $H^k(U \times (\mathbb{R}^r - \{0\})) =$

$$\underbrace{H^k(U)}_{i=k} \oplus \underset{i+(r-1)=k}{H^{k-(r-1)}(U)}$$

$i=k$ $\qquad\qquad\qquad\qquad$ $i+(r-1)=k$

Exact sequence of a pair
$$(U\times\mathbb{R}^r,\ U\times(\mathbb{R}^r-\{0\}))$$
and know $H^k(U\times\mathbb{R}^r)=H^k(U)$

$\Longrightarrow$ $H^k(U)$ cancels out in the exact sequence, so

$U\times\mathbb{R}^r - U$

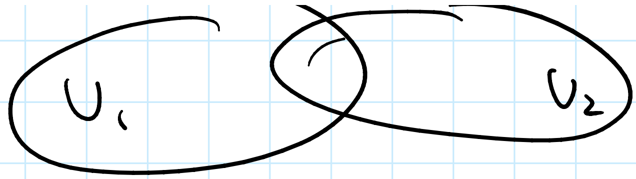
$$\boxed{H^k(U\times\mathbb{R}^r;\ U\times(\mathbb{R}^r-\{0\}))} =$$
$$= H^{k-1}(U\times\{\mathbb{R}^r-0\})\Big/H^{k-1}(U)$$
$$= H^{k-1-(r-1)}(U) = \underline{H^{k-r}(U)}.$$

To glue the neighborhoods where V is trivialized, use Mayer-Vietoris and check that the isomorphisms

zero section

$$H^i(U)\simeq H^{i+r}(\pi^{-1}(U);\pi^{-1}(U)\setminus U)$$
(*)

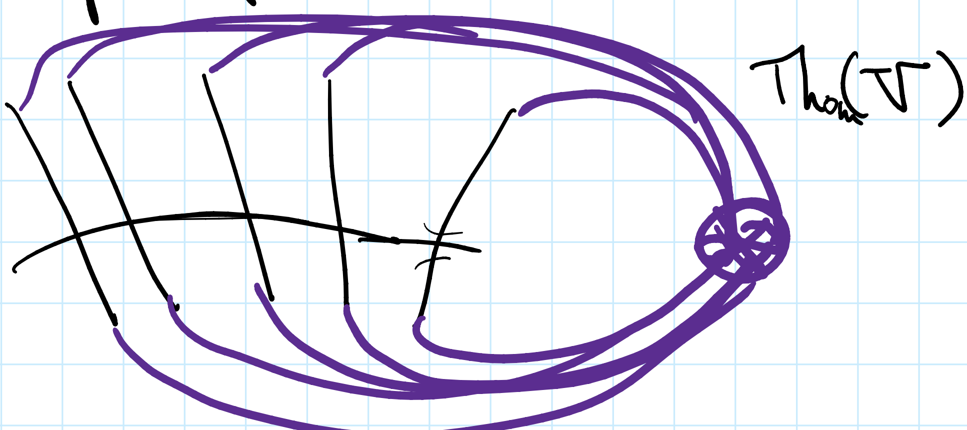
with it



If (*) holds for U_1 and U_2 and $U_1 \cap U_2$ then it holds for $U_1 \cup U_2$ and proceed by induction.

Two more rephrasings of Thom isomorphism:

1) Thom space of V: one-point compactification of V



$$\tilde{H}^*(\mathrm{Thom}(V)) = H^*(\mathrm{Thom}(V); *)$$

point we added

excision

$$= H^*(V; V \setminus M)$$

Each fiber compactifies in Thom(V)

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to $S^r = \mathbb{R}^r \cup *$

$$S^r - \{0\} = \mathbb{R}^r \setminus \{0\} \cup *$$

retracts to $*$

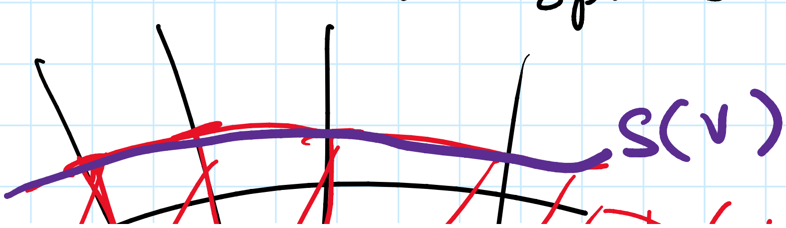
__Rmk__ Can use Thom space to reprove Thom isomorphism thm:
choose a cell decomposition of M such that the bundle is trivial on every cell, and check that

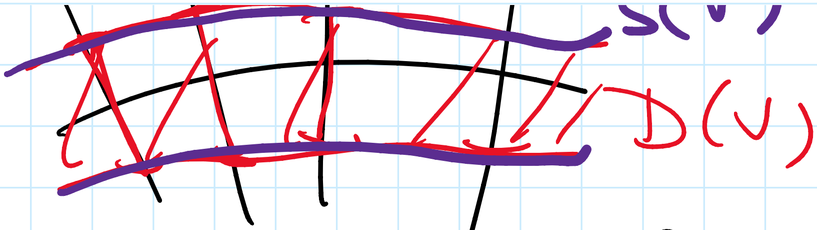
i-cell in M $\longleftrightarrow$ (i+r)-cell in Thom(V)

2) Choose a metric on V
(continuous quad. form on every fiber)

$D(V)$ = bundle of unit disks in V

$S(V) = \partial D(V)$ = bundle of unit spheres in V





fibers of $D(V) = \mathbb{D}^r$

fiber of $S(V) = S^{r-1}$

$$H^*(V; V \setminus M) = H^*(\mathrm{Thom}(V); \otimes)$$

$$\cong H^*(D(V); S(V))$$

exercise.

Construction of Euler class:

$$(V, \emptyset) \hookrightarrow (V; V - M)$$

$$\begin{array}{ccc} H^*(V; V-M) & \longrightarrow & H^*(V) \\ & & \| \\ & & H^*(M) \end{array}$$

Start from the class $1 \in H^0(M)$

$\longrightarrow$ by Thom isomorphism

$$\begin{array}{ccc} H^0(M) & = & H^r(V; V-M) \\ 1 & \longrightarrow & \tau = \text{Thom class} \end{array}$$

$\leftarrow$ choose a generator in $H^r(\mathbb{R}^r, \mathbb{R}^r - 0)$ in every fiber in every fiber

$$H^r(M) \simeq H^r(V) \ni e(V)$$

<u>Thm</u> This is equivalent to the construction from last lecture if M is smooth manifold, and $e(V)$ has the same general properties:

(a) $e(V)=0$ if V has a nonzero section

(b) $e(V_1 \oplus V_2) = e(V_1) \cup e(V_2)$

(c) $e(\text{trivial bdle}) = 0$

$$e(V \oplus \text{trivial}) = 0$$

(d) If we change orientation of the bundle, we $e(V) \leftrightarrow -e(V)$ change.

<u>Functoriality</u>

$$\begin{tikzcd} f^*V \arrow[r] \arrow[d] & V \arrow[d, "\pi"] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd}$$

top. spaces

Can always define a vector bundle f^*V over M_1

vector bundle f^*V over M_1

Fiber of f^*V over $x \in M_1$

$=$ fiber of V over $f(x)$

Alternatively,

$$f^*V = \{(x, v) : x \in M_1, v \in V \text{ and } f(x) = \pi(v)\}$$

Exercise f^*V is locally trivial

Ex $M_1 \subset M_2$ with $V \to M_2$
subset

$i : M_1 \longrightarrow M_2$ inclusion

$i^*V =$ restriction of V to M_1

this is a vector bundle on M_1

Thm $\boxed{e(f^*V) = f^*(e(V))}$

for any map $f : M_1 \to M_2$

$e(V) \in H^r(M_2)$ $\qquad e(f^*V) \in H^r(M_1)$

Ex $\mathbb{C}P^n = \{\text{lines in } \mathbb{C}^{n+1}\}$

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line $\ell \longleftrightarrow p_\ell \in \mathbb{C}P^n$

$\mathcal{O}(k)$ = complex vector bundle over $\mathbb{C}P^n$
rank 1

$k \geq 0$: fiber of $\mathcal{O}(k)$ over p_ℓ
= space of degree k homog. polynomials on line ℓ

Q: How to compute $e(\mathcal{O}(k))$?

$\mathcal{O}(k)$ = complex rank 1 bundle
$\Rightarrow$ real orientable rank 2 bundle

$e(\mathcal{O}(k)) \in H^2(\mathbb{C}P^n)$

Solution 1 Choose a section of $\mathcal{O}(k)$
look at its zero locus $Z \subset \mathbb{C}P^n$
compute $[Z]$, apply PD

Solution 2 $\mathbb{C}P^1 \xhookrightarrow{i} \mathbb{C}P^n$

$\mathbb{C}^2 \subset \mathbb{C}^{n+1}$

$\underline{\mathbb{C}} \subset \mathbb{C}$

<u>Claim</u>: restriction of $\mathcal{O}(k)$ on $\mathbb{CP}^n$ to $\mathbb{CP}^1 = \mathcal{O}(k)$ on $\mathbb{CP}^1$

$$i^*\mathcal{O}(k) = \mathcal{O}(k)$$

By functoriality,

$$\underset{H^2(\mathbb{CP}^n)}{\underset{\cap}{i^* e(\mathcal{O}(k))}} = \underset{H^2(\mathbb{CP}^1) = \mathbb{Z}}{\underset{\cap}{e(\mathcal{O}(k))}}$$

And $i^* : H^2(\mathbb{CP}^n) \longrightarrow H^2(\mathbb{CP}^1)$ is an iso

Therefore it is sufficient to compute $e(\mathcal{O}(k))$ on $\mathbb{CP}^1$

↑ real orientable rank 2 bundle on $\mathbb{CP}^1$ (S^2)

For that, choose a generic section and count its zeros

Because everything is complex, all zeros appear with positive signs.