Lecture 28 (6/2)

Wednesday, June 2, 2021 2:10 PM

Last time: Euler class

$V \to M$ rank r oriented vector bundle

$\rightsquigarrow e(V) \in H^r(M;\mathbb{Z})$

if V is not orientable, can still define $e(V) \in H^r(M;\mathbb{Z}_2)$

Properties:

- **functoriality**
$$\begin{tikzcd} f^*V \arrow[d] & V \arrow[d] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd}$$
$$e(f^*V) = f^*(e(V))$$
- **(Whitney) sum formula** $e(V_1 \oplus V_2) = e(V_1) \cup e(V_2)$
- If V has a nonzero section then $e(V) = 0$.

Rmk If V has a nonzero section s then it has a rank 1 trivial subbundle $\mathrm{Span}(s) \subset V$

$U = V/\mathrm{Span}(s)$ and

$$U = V/\mathrm{Span}(s) \quad \text{and}$$

$$\mathrm{Span}(s) \oplus U = V$$

Choose a metric....

$$e(V) = \underbrace{e(\mathrm{Span}(s))}_{=0} \cup e(U) = 0$$

<u>Rmk</u> In fact, sum formula works in greater generality

$A \subset V$ subbundle

$B = V/A$ then

$$e(V) = e(A) \cup e(B)$$

This works in many other setups e.g. algebraic geometry.

Characteristic classes

Two cases: $\quad V \longrightarrow M$

(A) Real rank r vector bundle V

(B) Complex rank r vector bundle V

__Thm__ There exists a unique collection of classes:

(A) $w_0 = 1 \in H^0(M;\mathbb{Z}_2), w_1 \in H^1(M;\mathbb{Z}_2) \ldots w_r \in H^r(M;\mathbb{Z}_2)$

w_i = Stiefel-Whitney classes

(B) $c_0 = 1 \in H^0(M); c_1 \in H^2(M), \ldots c_r \in H^{2r}(M)$

c_i = Chern classes. only even cohomology!

Note: if V is a complex rank r bundle, it is a real orientable rank $2r$ bundle.

Satisfying the following axioms:

(1) __Functoriality__ $w_i(f^*V) = f^* w_i(V)$

$c_i(f^*V) = f^* c_i(V)$

(2) __Sum formula__ (works for subbundles & quotients).

$$w(V_1 \oplus V_2) = w(V_1) \cup w(V_2)$$

$$c(V_1 \oplus V_2) = c(V_1) \cup c(V_2)$$

Where $c(V) = c_0 + c_1 + \ldots + c_r$ = full Chern class

where $c(V) = c_0 + c_1 + \dots + c_r$ = full Chern class
= formal sum of Chern classes.

$$(\Leftarrow) \quad \boxed{c_k(V_1 \oplus V_2) = \sum_{i=0}^{k} c_i(V_1) \cup c_{k-i}(V_2)}$$

And the same for w_i.

(3) Relation to the Euler class

(A) $w_r(V) = e(V) \mod 2$

(B) $c_r(V) = e(V)$

Fact If V is trivial then $c_i(V) = 0$

except $c_0 = 1$.

work with c_i
everything is similar for w_i

Cor Suppose that V has a trivial rank 1 subbundle

($\Leftarrow$) nonzero section s)

$V = \mathrm{Span}(s) \oplus \tilde{V}$ ← rank $(r-1)$ bdle

$$c(V) = \underset{\substack{\shortparallel \\ 1}}{c(\mathrm{Span}(s))} \cup c(\tilde{V})$$

$$= c_0(U) + c_1(U) + \dots + c_{r-1}(U)$$

$$\begin{cases} c_i(V) = c_i(U) & i \leq r-1 \\ c_r(V) = 0 \leftarrow \text{Euler class} \end{cases}$$

Ex If V has 2 linearly independent sections $\Longleftrightarrow$ U has a nonzero section

we can repeat this!

$$\Rightarrow c_{r-1}(U) = 0 = c_{r-1}(V) = 0.$$

Ex If V has k linearly independent sections $\Leftrightarrow$ rank k trivial subbundle $\Rightarrow V = \mathbb{R}^k \oplus U$

$$c(V) = c(\mathbb{R}^k) \cup c(U)$$

$$c(V) = c(U) = 1 + \dots + c_{r-k}(U)$$

$$\Rightarrow \boxed{c_r(V) = c_{r-1}(V) = \dots = c_{r-k+1}(V) = 0}$$

Rmk $c_r(V) \neq 0 \Rightarrow$ no nonzero

Rmk $\underset{\parallel\, e(V)}{c_r(V)} \neq 0 \Rightarrow$ no nonzero sections

$c_r(V) = 0,\ c_{r-1}(V) \neq 0 \Rightarrow$ may have at most 1 indep. section

$c_r(V) = 0,\ c_{r-1}(V) = 0,\ c_{r-2}(V) \neq 0$

$\Rightarrow$ at most 2 indep. sections.

And so on.

Ex L = complex line bundle (complex rank 1 bundle)

$$c(L) = 1 + c_1(L)$$

$$c_1(L) = e(L) \in H^2(L)$$

Ex $V = L_1 \oplus \dots \oplus L_r$

direct sum of r line bundles

$$x_i = c_1(L_i)$$

$$c(V) = c(L_1) \cup \dots \cup c(L_r)$$

$$= (1+x_1)(1+x_2) \dots (1+x_r)$$

$c_k(V)$ = k-th elementary

$c_k(V) = k$-th elementary symmetric function in x_i

$c_1(V) = \sum x_i$

$c_2(V) = \sum_{i<j} x_i x_j \quad \ldots$

<u>Thm</u> (Splitting principle)
Suppose that $V \longrightarrow M$ is a complex rank r vector bundle.
Then there exists another space $X \xrightarrow{f} M$ such that:

(1) $f^*: H^*(M) \longrightarrow H^*(X)$ is injective.

(2) $f^*V = L_1 \oplus \ldots \oplus L_r$ for some line bundles $L_1 \ldots L_r$ on X.

<u>Cor</u> $c_k(f^*V) = $ elem. symm. function in $c_1(L_i) \leftarrow$ can define/compute

$$c_k(f^*V) = f^*(c_k(V))$$

$f^*(c_k(V))$

Since f^* is injective, we do not lose information.

Ex $\mathbb{CP}^1$ has a tautological line bundle L

fiber of L over $p_\ell \longleftrightarrow$ line ℓ

point in $\mathbb{CP}^1$ representing line ℓ

$\mathcal{O}(k)$ = degree k polynomial on ℓ

$\mathcal{O}(1)$ = degree 1 polynomial on ℓ
$= L^*$

$c_1(\mathcal{O}(1)) = ?$ Choose a linear function on $\mathbb{C}^2$

$s = ax + by$

This defines a linear function on each line $\ell \Rightarrow$ section of $\mathcal{O}(1)$

$s = 0 \Longleftrightarrow ax+by$ vanishes on ℓ

$\Longleftrightarrow \ell = \{ax+by=0\}$

This is one line in $\mathbb{C}^2 \Longrightarrow$

This is one line in $\mathbb{C}^2$;

one point in $\mathbb{C}P^1$

$$\{s=0\} = pt \xleftrightarrow{PD} c_1(\mathcal{O}(1)) = a = \text{generator of } H^2(\mathbb{C}P^1)$$

$$c_1(\mathcal{O}(1)) = e(\mathcal{O}(1))$$

$$c(\mathcal{O}(1)) = 1 + a$$

__Rmk__ Can do the same for $\mathcal{O}(k)$ (HW)

s = generic degree k homogeneous polynomial in x and y.

This defines a homogeneous degree k polynomial on each line ℓ

$\Rightarrow$ section of $\mathcal{O}(k)$.

HW Need to compute # zeros of this section.

__Ex__ Same for $\mathbb{C}P^n$

$$\mathbb{C}P^1 \xhookrightarrow{f} \mathbb{C}P^n$$

$$\mathcal{O}(1)$$

$$f^*\mathcal{O}(k) = \mathcal{O}(k) \text{ on } \mathbb{C}P^1$$

$$c_1(f^*\mathcal{O}(1)) = f^*c_1(\mathcal{O}(1))$$

$$c_1(f^*\mathcal{O}(1)) = f^*\underbrace{c_1(\mathcal{O}(1))}$$

$$\begin{array}{c} \shortparallel \\ 1 + a \end{array} \qquad f^* \text{ is iso on } H^0, H^2$$

$$\Rightarrow \boxed{c_1(\mathcal{O}(1)) = 1 + a} \text{ on } \mathbb{C}P^n$$

<u>Ex</u> L = taut. bundle = dual to $\mathcal{O}(1)$

$$c(L) = 1 - a$$

Want to read more?

Milnor and Stasheff. "Char. classes"

Bott and Tu. Diff. forms ...

Hatcher "K-theory and vect bdles"