Lecture 3 (4/2)

Friday, April 2, 2021 2:03 PM

Cup product in cohomology

R = commutative ring
(in practice, $\mathbb{Z}$ or a field)

$C^k(X;R)$ = singular/simplicial cochains on X with coeffs in R

$\varphi \in C^k(X;R)$ $\psi \in C^\ell(X;R)$

Want to define their product

$$\varphi \cup \psi \in C^{k+\ell}(X;R)$$

Def $\sigma : \Delta^{k+\ell} \longrightarrow X$ singular simplex with vertices $v_0, \dots v_{k+\ell}$

$$\varphi \cup \psi(\sigma) = \varphi(\sigma[v_0 \dots v_k])\,\psi(\sigma[v_k \dots v_{k+\ell}])$$

k-dim simplex with these vertices

ℓ-dim simplex

Properties: ① R-bilinear

$$(\varphi+\varphi') \cup \psi = \varphi \cup \psi + \varphi' \cup \psi$$

$$(\varphi+\varphi')\cup\psi = \varphi\cup\psi + \varphi'\cup\psi$$

② Associative $(\varphi\cup\psi)\cup\eta = \varphi\cup(\psi\cup\eta)$
(clear)

③ $\boxed{\delta(\varphi\cup\psi) = \delta\varphi\cup\psi + (-1)^k\varphi\cup\delta\psi}$

(Leibniz rule) similar to the product rule for derivatives.

Proof: $\sigma : \Delta^{k+\ell+1} \to X$

$$\delta(\varphi\cup\psi)(\sigma) \overset{(\delta \text{ and } \partial \text{ are dual})}{=} \varphi\cup\psi(\partial\sigma) \overset{\text{def of } \partial\sigma}{=}$$

$$= \varphi\cup\psi\left(\sum_{i=0}^{k}(-1)^i\sigma[v_0,\dots\hat{v}_i,\dots v_{k+\ell+1}]\right) +$$

$$+ \varphi\cup\psi\left(\sum_{i=k+1}^{k+\ell+1}(-1)^i\sigma([v_0\dots\hat{v}_i\dots v_{k+\ell+1}]\right)$$

def of cup product

$$= \sum_{i=0}^{k}(-1)^i\varphi(\sigma[v_0,\dots\hat{v}_i\dots v_{k+1}])\psi(\sigma[v_{k+1}\dots v_{k+\ell+1}])$$

$$+ \sum_{i=k+1}^{k+\ell+1}(-1)^i\varphi(\sigma[v_0,\dots v_k])\psi(\sigma[v_k\dots\hat{v}_i\dots v_{k+\ell+1}])$$

defn of ∂

$$= \varphi(\partial\sigma[v_0,\dots v_{k+1}])\psi(\sigma[v_{k+1}\dots v_{k+\ell+1}])$$

$$+ (-1)^k\varphi(\sigma[v_0\dots v_k])\psi(\partial\sigma[v_k\dots v_{k+\ell+1}])$$

∂ and δ are dual

$$= \delta\varphi(\sigma[v_0\dots v_{k+1}])\psi(\sigma[v_{k+1}\dots v_{k+\ell+1}])$$

$$+ (-1)^k \varphi(\sigma[v_0 \ldots v_k])\, \delta\psi(\sigma[v_k \ldots v_{k+l+1}])$$

def of cup product

$$= \delta\varphi \cup \psi(\sigma) + (-1)^k \varphi \cup \delta\psi(\sigma)$$

---

(4) $\cup$ defines a product in cohomology!

$$H^k(X;R) \times H^l(X;R) \longrightarrow H^{k+l}(X;R)$$

Construction: pick representative cocycles, apply $\cup$, need to check that this is well defined!

(1) $\delta\varphi = \delta\psi = 0$ (φ, ψ = cocycles)

$$\delta(\varphi \cup \psi) = \cancel{\delta\varphi} \cup \psi + (-1)^k \varphi \cup \cancel{\delta\psi}$$

$$= 0 \Rightarrow \varphi \cup \psi \text{ is also a cocycle.}$$

(2) $\delta\varphi = 0$, $\tilde{\psi} = \psi + \delta\beta$

Then $\varphi \cup \tilde{\psi} = \varphi \cup \psi + \varphi \cup \delta\beta$

On the other hand,

$$\delta(\varphi \cup \beta) = \cancel{\delta\varphi} \cup \beta \pm \underline{\underline{\varphi \cup \delta\beta}}$$

this is coboundary!

$$\Rightarrow [\varphi \cup \tilde{\psi}] = [\varphi \cup \psi]$$

$\Rightarrow [\varphi \cup \tilde{\varphi}] = [\varphi \cup \psi]$.

$(C^\bullet, \delta)$ is an example of a differential graded algebra (dga)
= graded algebra + δ satisfying Leibniz rule.

We proved that (1) Cochains on X is a dga
(2) $H^*(C^\bullet, \delta)$ is an algebra for any dga $(C^\bullet, \delta)$.

---

⑤ Unit: $1 \in C^0(X; \mathbb{R})$

cochain which has value 1 on any 0-simplex

$$1 \cup \psi\ (\sigma) = 1(\sigma[v_0]) \cdot \psi(\sigma[v_0 \ldots v_k])$$
$$= \psi(\sigma).$$

$\delta 1 = 0 \Rightarrow$ this defines a unit in H^0.

---

⑥ Naturality $f: X \to Y$ continuous map

$$\boxed{f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)}$$

$$\boxed{f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)}$$

so $f^*: C^\bullet(Y) \longrightarrow C^\bullet(X)$

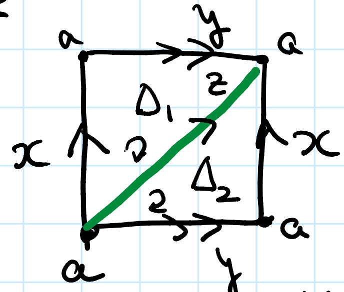
$\qquad H^\bullet(Y) \longrightarrow H^\bullet(X)$

is a (dga) homomorphism of rings.

Proof: $\Delta^{k+l} \xrightarrow{\sigma} X \xrightarrow{f} Y$

$$f^*(\alpha \cup \beta)(\sigma) = \alpha \cup \beta (f\sigma) =$$
$$= \alpha(f\sigma[v_0, \dots v_k])\,\beta(f\sigma[v_k \dots v_{k+l}])$$
$$= f^*\alpha(\sigma[v_0 \dots v_k])\,f^*\beta(\sigma[v_k, \dots v_{k+l}])$$
$$= [f^*\alpha \cup f^*\beta](\sigma). \quad \blacksquare$$

Ex T^2 torus



$$\partial(x) = \partial(y) = \partial(z) = 0$$
$$\partial(\Delta_1) = x + y - z$$
$$\partial(\Delta_2) = z - x - y$$

$H_0 = \mathbb{Z}\langle a\rangle$

$H_1 = \mathbb{Z}\langle x, y\rangle \qquad z = x + y$

$H_2 = \mathbb{Z}\langle \Delta_1 + \Delta_2\rangle$

By universal coefficient thm we get:

By universal coefficients thm

we get:

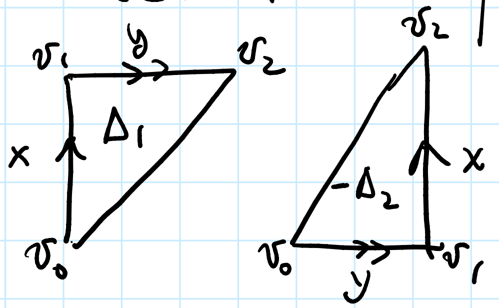
$$H^0 = \mathbb{Z} \qquad H^1 = \underset{\alpha}{\mathbb{Z}} \oplus \underset{\beta}{\mathbb{Z}} \qquad H^2 = \overset{\psi}{\mathbb{Z}}$$

$\psi(\Delta_1 + \Delta_2) = 1$

determined by the value on $\Delta_1 + \Delta_2$

Can choose α and β such that

$$\left.\begin{array}{ll} \alpha(x) = 1 & \beta(x) = 0 \\ \alpha(y) = 0 & \beta(y) = 1 \\ \alpha(z) = 1 & \beta(z) = 1 \end{array}\right\} \text{ dual basis to } \langle x, y\rangle = H_1$$



$\alpha \cup \alpha (\Delta_1) = \alpha(x)\alpha(y) = 0$

$\alpha \cup \alpha (-\Delta_2) = \alpha(y)\alpha(x) = 0$

$\Rightarrow \alpha \cup \alpha = 0$

Similarly, $\beta \cup \beta = 0$

$\alpha \cup \beta (\Delta_1) = \alpha(x)\beta(y) = 1$

$\alpha \cup \beta (-\Delta_2) = \alpha(y)\beta(x) = 0$

$\alpha \cup \beta (\Delta_1 + \Delta_2) = 1 - 0 = 1 \Rightarrow \alpha \cup \beta = \psi.$

$\beta \cup \alpha (\Delta_1) = \beta(x)\alpha(y) = 0$

$\beta \cup \alpha (-\Delta_2) = \beta(y)\alpha(x) = 1$

$\beta \cup \alpha (\Delta_1 + \Delta_2) = 0 - 1 = -1 = \alpha \cup \beta = -\psi.$

<u>Conclude</u>: $H^*(T^2) = \langle \underset{H^0}{1}, \underbrace{\alpha, \beta}_{H^1}, \underbrace{\psi}_{H^2} \rangle$

$\alpha \cup \beta = \psi$

$\beta \cup \alpha = -\psi \qquad \alpha \cup \psi \in H^3 = 0 \Rightarrow$ no other

$H^0 \quad H^1 \quad H^2$

$\alpha \cup \beta = \psi$

$\beta \cup \alpha = -\psi$ $\quad \alpha \cup \psi \in H^3 = 0 \Rightarrow$ no other interesting products by degree reasons.

Note: This is exterior algebra in α, β.

Thm In cohomology, cup product is graded (super) commutative

$$\boxed{\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha}$$

for $\alpha \in H^k(X)$ and $\beta \in H^l(X)$.

Note: This is false on cochains!

In the above example, $\deg(\alpha) = \deg(\beta) = 1$

$$\alpha \cup \beta = (-1)^{1 \cdot 1} \beta \cup \alpha = -\beta \cup \alpha.$$

Ex k or l are even $\Rightarrow \alpha \cup \beta = \beta \cup \alpha$

Both k, l are odd $\Rightarrow \alpha \cup \beta = -\beta \cup \alpha$.

Ex If $\alpha \in H^k$, k odd

$$\alpha \cup \alpha = (-1)^{k^2} \alpha \cup \alpha = -\alpha \cup \alpha$$

$$\boxed{2 \alpha \cup \alpha = 0}$$

If we work over $\mathbb{R}$ or $\mathbb{Q} \Rightarrow \alpha \cup \alpha = 0$

If we work over $\mathbb{Z}$ $\Rightarrow$ $\alpha \cup \alpha$ is a 2-torsion

If we work over $\mathbb{Z}_2$ $\Rightarrow$ no restriction (and H^* is commutative).