Lecture 4 (4/5)

Monday, April 5, 2021 2:07 PM

Thm $\alpha \in H^k(X)$ $\beta \in H^\ell(X)$ coeffs in any commutative ring R

$$\alpha \cup \beta = (-1)^{k\ell} \beta \cup \alpha \quad \text{in } H^{k+\ell}(X)$$

Morally: $\alpha \cup \beta = (-1)^{k\ell} \beta \cup \alpha + \delta(\cdots)$

Proof: $\sigma : [v_0 \ldots v_n] \longrightarrow X$ simplex

$\bar{\sigma} : [v_n \ldots v_0] \longrightarrow X$ reverse order of vertices

$\bar{\sigma}(v_i) = v_{n-i}$

Change orientation by a permutation

$$\operatorname{sgn}\begin{pmatrix} 0 & \ldots & n \\ n & \ldots & 0 \end{pmatrix} = (-1)^{\frac{n(n+1)}{2}} = \varepsilon_n$$

every pair (i,j) is an inversion

Define $\rho(\sigma) = (-1)^{\frac{n(n+1)}{2}} \bar{\sigma} = \varepsilon_n \bar{\sigma}$.

Lemma ① ρ is a chain map $\partial \rho = \rho \partial$

② ρ is chain homotopic to identity

$\rho - \mathbb{1} = \partial P + P \partial$ for some P.

③ Cor from ②: $\rho = \mathbb{1}$ on $H_*(X)$.

Proof of lemma ① $\partial \rho(\sigma) = \varepsilon_n \sum (-1)^i \sigma[v_n \ldots \hat{v} \ldots v_0]$

Proof of Lemma ① $\partial\rho(\sigma) = \varepsilon_n \sum_i (-1)^i \sigma[v_n, \dots \hat{v}_{n-i} \dots v_0]$

$$\rho(\partial\sigma) = \rho\left(\sum (-1)^{n-i} \sigma[v_0 \dots \hat{v}_{n-i} \dots v_0]\right) =$$

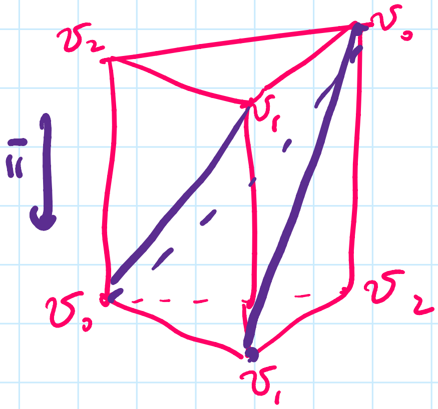
apply ρ to $(n-1)$ simplex

$$= \varepsilon_{n-1} \cdot \sum (-1)^{n-i} \sigma[v_n \dots \hat{v}_{n-i} \dots v_0]$$

Note: $\underset{(-1)^{\frac{n(n+1)}{2}}}{\varepsilon_n} = (-1)^n \cdot \underset{(-1)^{\frac{n(n-1)}{2}}}{\varepsilon_{n-1}} \Rightarrow \partial\rho = \rho\partial$, so ρ is a chain map.

② $P(\sigma) = \sum_i (-1)^i \varepsilon_{n-i}\, \sigma[v_0 \dots v_i, v_n \dots v_i]$

singular $(n+1)$-simplex. projects to n-simplex then maps to X



$i=0$ $\underset{\text{bot}}{v_0}, \overline{v_2, v_1, v_0}^{\text{top}}$

$i=1$ $\underbrace{v_0, v_1}_{\text{bot}}, \underbrace{v_2, v_1}_{\text{top}}$

$i=2$ v_0, v_1, v_2, v_0

$$\partial P(\sigma) = \sum_{j \le i} (-1)^i (-1)^j \varepsilon_{n-i}\, \sigma[v_0 \dots \hat{v}_j \dots v_i, v_n \dots v_i] +$$

$$+ \sum_{j \ge i} (-1)^i (-1)^{i+1+n-j} \varepsilon_{n-i}\, \sigma[\overbrace{v_0 \dots v_i}^{i+1}, \overbrace{v_n \dots}^{n-j} \hat{v}_j \dots v_i]$$

$$P(\partial\sigma) = \sum_{i<j} (-1)^i (-1)^j \varepsilon_{n-i-1}\, \sigma[v_0 \dots v_i, v_n \dots \hat{v}_j \dots v_i]$$

$$+ \sum (-1)^{i-1} (-1)^j \varepsilon \dots \sigma[v \dots \hat{v} \dots v_i]$$

$$+\sum_{i>j}(-1)^{i-1}(-1)^{j}\varepsilon_{n-i}\,\sigma[v_0 \ldots \hat{v}_j \ldots v_i, v_n \ldots v_i]$$

Explicit check: for $i \neq j$ $\partial P + P\partial$ all terms will cancel out

For $i=j$, we get:

$$\underbrace{\varepsilon_n \sigma[v_n \ldots v_0]}_{\rho(\sigma)} + \sum_{i>0}\varepsilon_{n-i}\sigma[v_0 \ldots v_{i-1}, v_n \ldots v_i]$$

cancel out

$$+\sum (-1)^{n+i+1}\varepsilon_{n-i}\sigma[v_0 \ldots v_i, v_n \ldots v_{i+1}]$$ relabel $i \Leftrightarrow i+1$

$$-\underbrace{\sigma[v_0 \ldots v_n]}_{\sigma} = \rho(\sigma) - \sigma.$$

Conclusion: $\rho(\sigma) - \sigma = \partial P + P\partial$. ■.

<u>Proof of Thm from Lemma:</u> $\rho = 1$ in H_*

$\rho^* = 1$ in H^*

Recall cup product

$$\alpha \cup \beta\,(\sigma[v_0 \ldots v_{k+\ell}]) = \alpha(\sigma[v_0 \ldots v_k])\cdot$$
$$\cdot\beta(\sigma[v_k \ldots v_{k+\ell}])$$

$$(\rho^*\alpha \cup \rho^*\beta)(\sigma) = \varepsilon_k \cdot \varepsilon_\ell \cdot \alpha(\sigma[v_k \ldots v_0])\cdot$$
$$\cdot\beta(\sigma[v_{k+\ell}, \ldots v_k])$$

reverse order

$\rho^*(\alpha \cup \beta)(\sigma) =$

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$$= \varepsilon_{k+\ell} \cdot (\beta \cup \alpha)(\sigma[v_{k+\ell}, \dots v_0])$$

$$= \varepsilon_{k+\ell} \cdot \beta(\sigma[v_{k+\ell} \dots v_k]) \cdot \alpha([v_k \dots v_0])$$

reverse order separately inside α and β

<u>Conclusion</u>: $(\rho^*\alpha \cup \rho^*\beta) = (-1)^{k\ell} \rho^*(\beta \cup \alpha)$

On the other hand $\quad \varepsilon_{k+\ell} = (-1)^{k\ell} \varepsilon_k \varepsilon_\ell$

$\rho^*\alpha = \alpha$ in H^* $\quad \dfrac{(k+\ell)(k+\ell+1)}{2} = k\ell + \dfrac{k(k+1)}{2} + \dfrac{\ell(\ell+1)}{2}$.

$\rho^*\beta = \beta \qquad \rho^*(\beta \cup \alpha) = \beta \cup \alpha$.

For top. applications, proof is not that important

$$\boxed{\alpha \cup \beta = (-1)^{k\ell} \beta \cup \alpha \quad \text{in } H^*}$$

But the result is <u>extremely</u> important!

<u>Facts</u> (proof later)

① $H^*(\mathbb{RP}^n, \mathbb{Z}_2) = \dfrac{\mathbb{Z}_2[\alpha]}{(\alpha^{n+1} = 0)} \qquad \alpha \in H^1(\mathbb{RP}^n, \mathbb{Z}_2)$

Basis : $\underset{H^0}{1}, \underset{H^1}{\alpha}, \underset{H^2}{\alpha^2}, \underset{H^3}{\alpha^3}, \dots \alpha^n$

Recall: $H^i(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}_2$ ...

Recall: $H^i(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}_2 \quad i=0,\dots n$

$H^0 \quad H^1 \quad H^2 \quad H^3 \quad \dots$

hard prove later

spanned over $\mathbb{Z}_2$ by α^i

(differential $=0 \mod 2$) on cellular complex,

(2) $H^*(\mathbb{CP}^n, \mathbb{Z}) = \dfrac{\mathbb{Z}[\beta]}{(\beta^{n+1}=0)} \quad \beta \in H^2(\mathbb{CP}^n, \mathbb{Z})$

Basis: $1, \beta, \beta^2, \dots \beta^n$

Recall: $H^i(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z} \quad i=0,2,\dots 2n$

Span (β^i)

(differential $=0$ on cellular cpx)

<u>Rmk</u> $H^*(\mathbb{RP}^\infty, \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$

$H^*(\mathbb{CP}^\infty, \mathbb{Z}) = \mathbb{Z}[\beta]$

<u>Last lecture</u>: $H^*(T^2, \mathbb{Z}) = \langle 1, \underbrace{\alpha, \beta}_{H^1}, \underbrace{\alpha\beta = -\beta\alpha}_{H^2} \rangle$

exterior algebra with two generators α, β.

<u>Aside on tensor products</u>:

(1) A, B = vector spaces
w. bases $a_1 \dots a_n$, $b_1 \dots b_m$

$\Rightarrow A \otimes B$ with basis $a_i \otimes b_j$

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$$\dim A \otimes B = n \cdot m$$

Same for <u>free</u> abelian groups / free R-modules. (R commutative).

$$A \ni a = \sum \alpha_i a_i \qquad b = \sum \beta_i b_i \in B$$

$$a \otimes b = \left(\sum \alpha_i a_i\right) \otimes \left(\sum \beta_j b_j\right)$$

$$= \sum \alpha_i \beta_j \, a_i \otimes b_j$$

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(2)

$$\longrightarrow A_i \xrightarrow{\partial} A_{i-1} \xrightarrow{\partial} \cdots A_0$$

$$\longrightarrow B_i \xrightarrow{\partial} B_{i-1} \xrightarrow{\partial} \cdots B_0$$

$\Rightarrow$ tensor product of complexes (each of A_i, B_j = free module)

basis $\underbrace{a \otimes b}_{\text{degree } i+j}$ a, b = basis on A_0, B_0

$$\partial(a \otimes b) = \partial(a) \otimes b + (-1)^i a \otimes \partial b$$

if $a \in A_i$ $b \in B_j$
