Lecture 5 (4/7)

Wednesday, April 7, 2021 1:39 PM

$$\to A_i \xrightarrow{\partial} A_{i-1} \xrightarrow{\partial} \cdots \quad A_\bullet$$

$$\to B_j \xrightarrow{\partial} B_{j-1} \xrightarrow{\partial} \cdots \quad B_\bullet$$

complexes of vect. spaces / free abelian grps.

$$C_\bullet = A_\bullet \otimes B_\bullet$$

$$\boxed{\partial(a\otimes b) = \partial(a)\otimes b + (-1)^i a \otimes \partial(b)}$$

where $a \in A_i$

What does it mean?

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \\
\arrow[r] & A_i \otimes B_{j+1} \arrow[r] \arrow[d] & A_{i-1}\otimes B_{j+1} \arrow[r] \arrow[d] & {} \\
\arrow[r] & A_i \otimes B_j \arrow[r] \arrow[d] & A_{i-1}\otimes B_j \arrow[r] \arrow[d] & {} \\
\arrow[r] & A_i \otimes B_{j-1} \arrow[r] \arrow[d] & A_{i-1}\otimes B_{j-1} \arrow[r] \arrow[d] & {} \\
 & {} & {} &
\end{tikzcd}$$

C_{i+j}

C_{i+j-1}

$$C_k = \bigoplus_{i+j=k} A_i \otimes B_j$$

$i+j=k$

$$\begin{tikzcd} a\otimes b \arrow[r, "\partial"] \arrow[d, "\partial"'] & \partial(a)\otimes b \\ (-1)^i a\otimes\partial b & \end{tikzcd}$$

$$\partial : C_k \to C_{k-1}$$

Lemma $\partial^2 = 0$ on C.

Proof: $\partial^2(a\otimes b) = \partial(\underbrace{\partial(a)}_{i-1}\otimes b + (-1)^i a\otimes\partial(b))$

$= \cancel{\partial^2(a)\otimes b} + (-1)^{i-1}\partial(a)\otimes\partial(b) + (-1)^i\partial(a)\otimes\partial(b)$

$+ \cancel{(-1)^i\cdot(-1)^i\cdot a\otimes\partial^2(b)}$ ∎ cancel out.

Rmk

$$\partial^2 = \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & 0 \\ \bullet \arrow[r] \arrow[d] & \bullet & \\ 0 & & \end{tikzcd}$$

two terms cancel out.

Rmk $A_\bullet$ is a dg algebra (algebra + differential satisfies Leibniz rule)

$\iff$ chain map $A_\bullet\otimes A_\bullet \longrightarrow A_\bullet$

multiplication $a\otimes b \longrightarrow ab$

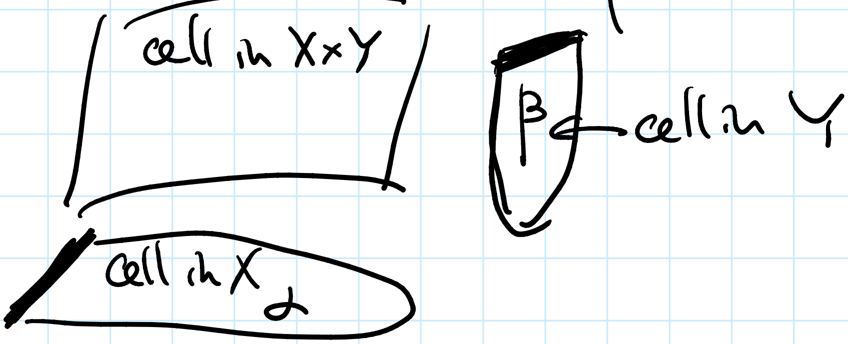
equivalent to Leibniz rule.

equivalent to Leibniz rule.

Ex: X = top. space, Y = top. space } CW complexes

$$(k\text{-cell in } X) \times (l\text{-cell in } Y) = (k+l\text{-cell in } X \times Y)$$

$\alpha \qquad \beta \qquad \alpha \times \beta$



Ex $X = Y = S^2$ $\qquad X = \bullet \sqcup 2\text{-cell}$, $\quad Y = \bullet \sqcup 2\text{-cell}$

$$S^2 \times S^2 = X \times Y = (\bullet \times \bullet) \sqcup (\bullet\, 2\text{-cell}) \sqcup (2\text{-cell} \times \bullet)$$
$$\sqcup (2\text{-cell} \times 2\text{-cell}).$$

$$\partial(\alpha \times \beta) = \partial\alpha \times \beta \cup \alpha \times \partial\beta$$

This is similar to our rule for $\partial(a \otimes b)$,

Sign is responsible for orientations.

$$\Rightarrow C_\bullet(X \times Y) = C_\bullet(X) \otimes C_\bullet(Y)$$

cellular chain complexes

$$C^\circ(X \times Y) = C^\circ(X) \otimes C^\circ(Y).$$

Thm Let $A_\cdot, B_\cdot$ = complexes over a field $\mathbb{K}$ of vect. spaces

Then $H_*(A_\cdot \otimes B_\cdot) = H_*(A) \otimes H_*(B)$

Cor $H_*(X \times Y; \mathbb{K}) = H_*(X;\mathbb{K}) \otimes H_*(Y;\mathbb{K})$

$H^*(X \times Y; \mathbb{K}) = H^*(X;\mathbb{K}) \otimes H^*(Y;\mathbb{K})$

Over a field $\mathbb{K}$

Künneth formula

Warning: This is false over $\mathbb{Z}$ in general

$H_*(\mathbb{R}P^2 \times \mathbb{R}P^2, \mathbb{Z}) \neq H_*(\mathbb{R}P^2) \otimes H_*(\mathbb{R}P^2)$

– see HW2 Idea: Compute $C_\cdot(\mathbb{R}P^2)$

$C_\cdot(\mathbb{R}P^2 \times \mathbb{R}P^2) = C_\cdot(\mathbb{R}P^2) \otimes C_\cdot(\mathbb{R}P^2)$

Proof: Break A and B into pieces

$0 \to \mathbb{K} \to 0 \qquad 0 \to \mathbb{K} \xrightarrow{1} \mathbb{K} \to 0$

4 cases: $[0 \to \mathbb{K} \to 0] \otimes [0 \to \mathbb{K} \to 0] =$

$H_* = \mathbb{K} \qquad H_* = \mathbb{K}$

(4 cases:) $[0 \to \mathbb{K} \to 0] \otimes [0 \to \mathbb{K} \to 0] =$

$H_* = \mathbb{K}$ $\qquad\qquad$ $H_* = \mathbb{K}$

$$\begin{tikzcd} & 0 \arrow[d] & \\ 0 \arrow[r] & \mathbb{K} \arrow[r] \arrow[d] & 0 \\ & 0 & \end{tikzcd} \quad = \quad 0 \to \mathbb{K} \to 0$$

$H_* = \mathbb{K}$

$\mathbb{K} \otimes_{\mathbb{K}} \mathbb{K} = \mathbb{K}$.

$[0 \to \mathbb{K} \to 0] \otimes [0 \to \mathbb{K} \xrightarrow{1} \mathbb{K} \to 0] =$

$H_* = \mathbb{K}$ $\qquad\qquad$ $H_* = 0$

$$\begin{tikzcd} & 0 \arrow[d] & \\ 0 \arrow[r] & \mathbb{K} \arrow[r] \arrow[d, "\pm 1"'] & 0 \\ 0 \arrow[r] & \mathbb{K} \arrow[r] \arrow[d] & 0 \\ & 0 & \end{tikzcd} \quad = \quad 0 \to \mathbb{K} \xrightarrow{\pm 1} \mathbb{K} \to 0$$

$H_* = 0$

$[0 \to \underset{i}{\underline{\mathbb{K}}} \xrightarrow{1} \mathbb{K} \to 0] \otimes [0 \to \mathbb{K} \xrightarrow{1} \mathbb{K} \to 0]$

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathbb{K} \arrow[r, "1"] \arrow[d, "(-1)^i"'] & \mathbb{K} \arrow[r] \arrow[d, "(-1)^{i-1}"] & 0 \\ 0 \arrow[r] & \mathbb{K} \arrow[r, "1"] \arrow[d] & \mathbb{K} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & \end{tikzcd}$$

$$0 \longrightarrow \mathbb{K} \xrightarrow{(1,\,(-1)^i)} \mathbb{K} \oplus \mathbb{K} \xrightarrow{((-1)^{i-1},\,1)} \mathbb{K} \longrightarrow 0$$

Can check that $H_* = 0$.

$$(A \oplus A') \otimes B = A \otimes B \oplus A' \otimes B$$

Since we've written A and B as direct sums of elementary pieces, and checked for $\otimes$ of those, we are done. $\blacksquare$

<u>Rmk</u> For any coefficient, we have a map $H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$

we checked that it is an isomorph over $\mathbb{K}$.

What about the ring structure?

<u>Thm</u> $H^*(X) \otimes_{\mathbb{K}} H^*(Y) \simeq H^*(X \times Y)$

as graded rings.

Generators $\underset{\substack{\shortparallel \\ a \otimes 1}}{a} \in H^k(X)$, $\underset{\substack{\shortparallel \\ 1 \otimes b}}{b} \in H^{\ell}(Y)$

All relations in $H^*(X)$ and $H^*(Y)$ hold for these

[illegible] $(-1)^{k\ell}$ [illegible]

+ sign rule $\boxed{a \cup b = (-1)^{kl}\, b \cup a}$

and no other relations.

<u>Ex</u> $T^2 = S^1 \times S^1$

$$H^*(S^1) = \langle 1, \alpha \rangle \qquad \alpha^2 = 0 \in H^2(S^1)$$

$$\alpha \in H^1(S^1)$$

$$H^*(S^1) = \langle 1, \beta \rangle \qquad \beta^2 = 0$$

$$\beta \in H^1(S^1)$$

$$H^*(T^2) = \underset{\alpha}{H^*(S^1)} \otimes \underset{\beta}{H^*(S^1)}$$

generated by α and β

relations $\alpha^2 = 0$, $\beta^2 = 0$

$$\alpha\beta = -\beta\alpha \qquad (-1)^{1\cdot 1} = -1$$

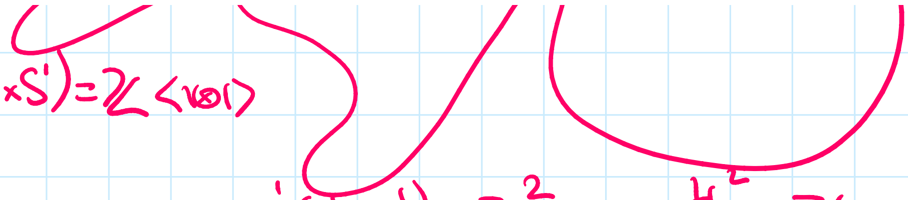
$\Rightarrow$ basis $\boxed{\underbrace{1}_{H^0}, \underbrace{\alpha, \beta}_{H^1}, \underbrace{\alpha\beta = -\beta\alpha}_{H^2}.}$

$$H^0(S^1) = \mathbb{Z} \qquad H^1(S^1) = \mathbb{Z}$$

$H^*(S^1) \otimes H^*(S^1)$: $H^0 \otimes H^0 = \mathbb{Z}$ $\quad H^0 \otimes H^1 = \mathbb{Z}$

$H^1 \otimes H^0 = \mathbb{Z}$ $\quad H^1 \otimes H^1 = \mathbb{Z}$

$H^0(S^1 \times S^1) = \mathbb{Z}\langle 1 \otimes 1 \rangle$



$H^0(S^1\times S^1)=\mathbb{Z}\langle 1\otimes 1\rangle$

$H^1(S^1\times S^1)=\mathbb{Z}^2$ $\langle\alpha,\beta\rangle$ $H^2=\mathbb{Z}$.

Rmk Künneth formula says

$$\boxed{H^k(X\times Y)=\bigoplus_{i+j=k} H^i(X)\otimes H^j(Y)}$$

Rmk If $H^*(X)$ or $H^*(Y)$ is free over $\mathbb{Z}$, Künneth formula still works.

Note: Our construction of cup product uses singular homology, and proof of Künneth formula uses cellular homology. One way to resolve it is to subdivide $\Delta^k\times\Delta^l$ into simplices (Eilenberg–Zilber)
