Lecture 6 (4/9)

Friday, April 9, 2021 2:08 PM

More on Künneth formula

Recall: $H^*(X\times Y) = H^*(X)\otimes H^*(Y)$ over a field.

① $X \xleftarrow{\pi_X} X\times Y \xrightarrow{\pi_Y} Y$ two projections

$a \leftarrow (a,b) \longrightarrow b$

$$\pi_X^*: \underset{\alpha}{H^*(X)} \longrightarrow H^*(X\times Y) = H^*(X)\otimes H^*(Y)$$

$$\pi_Y^*: \underset{\beta}{H^*(Y)} \longrightarrow H^*(X\times Y) = H^*(X)\otimes H^*(Y)$$

Fact: (a) $\pi_X^*(\alpha) = \alpha\otimes \underset{\in H^0(Y)}{1}$ in Künneth formula

(easy) $\pi_Y^*(\beta) = \underset{\in H^0(X)}{1}\otimes\beta$

$H^0(X\times Y) \ni 1 = 1\otimes 1$

Recall: Z = any top. space

$\Rightarrow \boxed{1\in H^0(Z)}$ $\quad 1(a) = 1$ on any 0-chain a

$\partial(x) = a - b$

$(\delta 1)(x) = 1(\partial x) = 1(a-b) =$

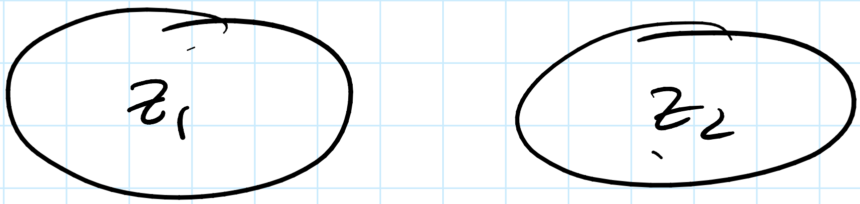
$= 1(a) - 1(b) = 0$

We ...

$= 1(a) - 1(b) = 0$

We proved earlier

$\alpha \cup 1 = 1 \cup \alpha = \alpha$ for any $\alpha \in H^k(Z)$



$Z = Z_1 \sqcup Z_2$ disjoint union

HW: compute $H^*(Z)$ in terms of $H^*(Z_1)$ and $H^*(Z_2)$

As a vector space $\boxed{H^*(Z) = H^*(Z_1) \oplus H^*(Z_2)}$

$C_*(Z) = C_*(Z_1) \oplus C_*(Z_2)$ for chain cpxes.

$H_*(Z) = H_*(Z_1) \oplus H_*(Z_2)$

For the HW problem, we know how to multiply classes in $H^*(Z_1)$ between themselves in $H^*(Z_2)$ between themselves.

Q: What happens is the compute cup product of a class in Z_1 and a class in Z_2?

$1_{Z_1} \in H^0(Z_1) \qquad 1_{Z_2} \in H^0(Z_2)$

$$1_{Z_1}(a) = \begin{cases} 1, \text{ if } a \text{ is a 0-cycle in } Z_1 \\ 0, \text{ if } a \text{ is a 0-cycle in } Z_2 \end{cases}$$

$$1_{Z_2}(a) = \begin{cases} 0, \text{ if } a \text{ is a 0-cycle in } Z_1 \\ 1 \text{ if } a \text{ is a 0-cycle in } Z_2. \end{cases}$$

$$\delta 1_{Z_1} = \delta 1_{Z_2} = 0$$

$$1_Z = 1_{Z_1} + 1_{Z_2}$$

(b) Note that π_X^* and π_Y^* are ring homomorphisms $\Rightarrow$ any relation between classes $\alpha \in H^*(X)$ also holds between classes $\xrightarrow{\pi_X^*} \alpha \otimes 1$ in $H^*(X \times Y)$

Same for $1 \otimes \beta$ where $\beta \in H^*(Y)$.

(c) (Harder) $\boxed{\alpha \otimes \beta = (\alpha \otimes 1) \cup (1 \otimes \beta)}$

$\alpha \in H^k(X) \qquad \beta \in H^l(Y)$

cup product in $X \times Y$

(d) $\boxed{(\alpha \otimes \beta) \cup (\gamma \otimes \varepsilon)} = ?$

$\alpha, \gamma \in H^*(X)$

$\beta, \varepsilon \in H^*(Y)$

(c) $\boxed{(\alpha\otimes\beta)\cup(\gamma\otimes\varepsilon)} = \,?$ $\qquad \alpha,\gamma\in H^*(X)$, $\beta,\varepsilon\in H^*(Y)$

$$\| $$

$$(\alpha\otimes 1)\cup(1\otimes\beta)\cup(\gamma\otimes 1)\cup(1\otimes\varepsilon) =$$

$$= (-1)^{\deg\beta\cdot\deg\gamma}\,(\alpha\otimes 1)\cup(\gamma\otimes 1)\cup(1\otimes\beta)\cup(1\otimes\varepsilon)$$

$$= (-1)^{\deg\beta\cdot\deg\gamma}\,((\alpha\cup\gamma)\otimes 1)\cup(1\otimes(\beta\cup\varepsilon))$$

$$= \boxed{(-1)^{\deg\beta\cdot\deg\gamma}\,(\alpha\cup\gamma)\otimes(\beta\cup\varepsilon)\,.}$$

__Ex:__ $S^1\times S^2$ $\qquad H^*(S^1)$: $1\in H^0$ $\quad \alpha\in H^1$

$\alpha(1\text{-cell})=1$

$\alpha^2=0\in H^2(S^1)$

$H^*(S^2)$: $1\in H^0$ $\quad \beta\in H^2$

$\beta^2=0\in H^4(S^2)$

$H^*(S^1\times S^2)=$ generated by $\alpha\otimes 1\sim\alpha$, $1\otimes\beta\sim\beta$

still have relations $\alpha^2=0$ $\quad \beta^2=0$

$$\alpha\cup\beta=(-1)^{1\cdot 2}\beta\cup\alpha=\beta\cup\alpha$$

Basis in $H^*(S^1\times S^2)$:

$$H^0=\langle 1\rangle \qquad H^1=\langle\alpha\rangle \qquad H^2=\langle\beta\rangle \qquad H^3=\langle\alpha\cup\beta=\beta\cup\alpha\rangle$$

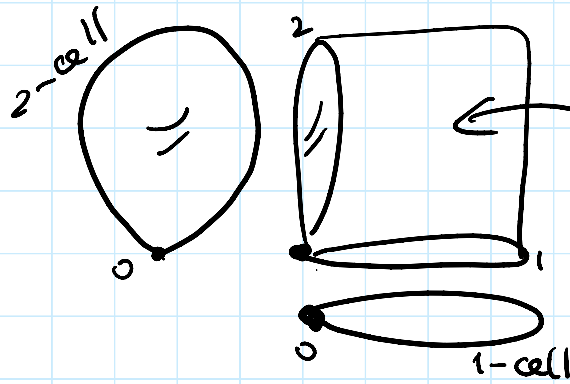
$$\quad\| \qquad\qquad \| \qquad\qquad \| \qquad\qquad \|$$

$$1\otimes 1 \qquad \alpha\otimes 1 \qquad 1\otimes\beta \qquad \alpha\otimes\beta$$

$\otimes 1 \qquad \alpha\otimes 1 \qquad 1\otimes\beta \qquad \alpha\otimes\beta$

$$\alpha\cup(\alpha\cup\beta) = \alpha^2\cup\beta = 0.$$

Cells in $S^1\times S^2$:



(1-cell) $\times$ (2-cell) $= 3$-cell $= D^3$.

$$\boxed{\partial(a\otimes b) = \partial(a)\otimes b + (-1)^i a\otimes\partial(b)}$$

$a = i$-cell in X $\qquad b = j$-cell in Y

(e) $X \xrightarrow{\Delta} X\times X$ diagonal map

$p \longrightarrow (p, p)$

$H^*(X\times X) = H^*(X)\otimes H^*(X)$ by Künneth

$\alpha\in H^*(X) \qquad \beta\in H^*(X)$

$\Delta^*: H^*(X\times X)\longrightarrow H^*(X)$

Claim $\boxed{\Delta^*(\alpha\otimes\beta) = \alpha\cup\beta}$

So Künneth formula and Δ^*

So Künneth formula and Δ^* give an <u>alternative definition of cup product</u>.

<u>Proof</u> $\Delta^*(\alpha\otimes\beta) = \Delta^*\big((\alpha\otimes 1)\cup(1\otimes\beta)\big) =$

(d) above

$= \Delta^*(\alpha\otimes 1)\cup\Delta^*(1\otimes\beta) =$

Δ^* is a ring homomorphism

$= \Delta^*(\pi_X^*(\alpha))\cup\Delta^*(\pi_Y^*(\beta))$

Observe that $\pi_X\circ\Delta = Id$

$$
\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[rr, bend right, "Id"'] & X\times X \arrow[r, "\pi_X"] & X \\
p \arrow[r] & (p,p) \arrow[r] & p
\end{tikzcd}
$$

$\pi_Y\circ\Delta = Id$

So $\Delta^*(\pi_X^*(\alpha)) = (\pi_X\circ\Delta)^*(\alpha) =$

$= (Id)^*(\alpha) = \alpha$.

Same for $\beta \Rightarrow$ we get $\alpha\cup\beta$. ∎

<u>Note</u>: In homology, $\Delta_*: H_*(X)\to H_*(X\times X)$ goes the other way and cannot

goes the other way and cannot
be used to define a product.