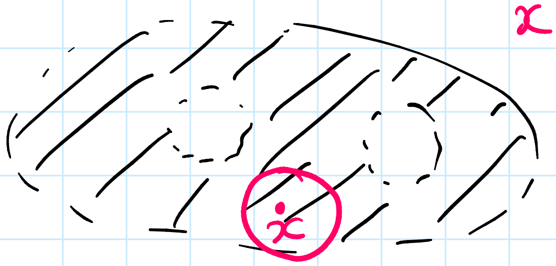
Lecture 7 (4/12)

Monday, April 12, 2021 1:46 PM

Manifolds

① A topological n-dimensional manifold X is a top. space which is **locally homeomorphic** to $\mathbb{R}^n$, that is, for any point $x \in X$ there is a open neighborhood $U \ni x$ such that U is homeomorphic to $\mathbb{R}^n$.

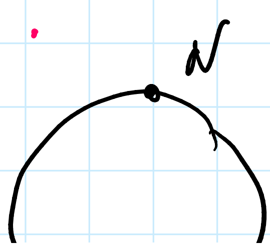
Ex Any open subset X in $\mathbb{R}^n$



$x \in X$, there is an open ball $U \subset X$ around x

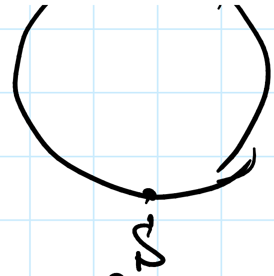
Any open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$.

Ex: S^n



$U_1 = S^n - \{N\} \simeq \mathbb{R}^n$

Ex: S^n $U_1 = S^n - \{N\} \simeq \mathbb{R}^n$



$U_2 = S^n - \{S\} \simeq \mathbb{R}^n$

$U_1 \cup U_2 = S^n$

Pick $x \in S^n$ if $x \neq N$ or S we can use either U_1 or U_2

$x = N$, can use U_2, if $x = S$, can use U_1

In practice, we will often assume that X is compact (closed n-manifold is compact, no boundary).

In this case, we can choose finitely many open charts $U_1 \dots U_N$ which are all homeomorphic to $\mathbb{R}^n$ and $\bigcup_{i=1}^{N} U_i = X$

Ex $S^n = U_1 \cup U_2$

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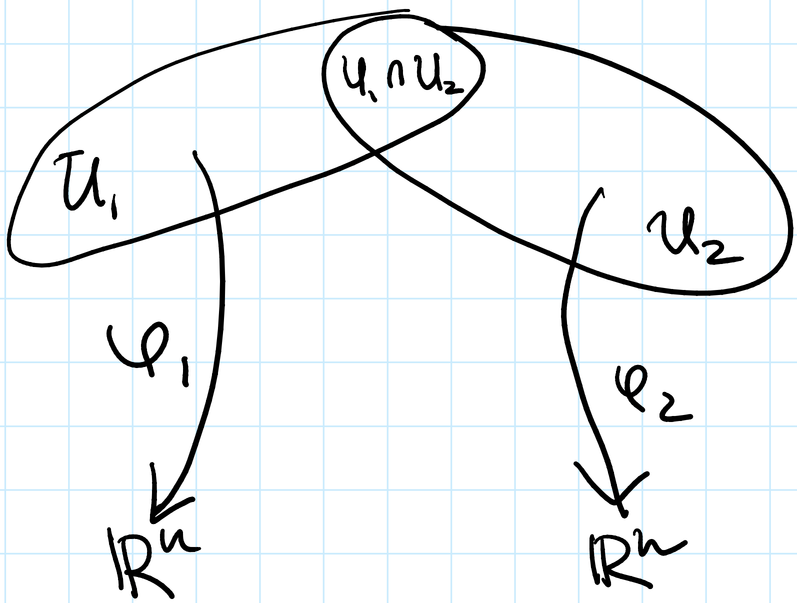
So we can use just two charts.

U_1, U_2 = two of these charts

$\varphi_1 : U_1 \longrightarrow \mathbb{R}^n$, $\varphi_2 : U_2 \longrightarrow \mathbb{R}^n$
homeomorphisms.



We can use φ_1 and φ_2 to identify $U_1 \cap U_2$ with two <u>different</u> open subsets in $\mathbb{R}^n$:

$$\begin{tikzcd} & \underbrace{U_1 \cap U_2}_{} \subset X \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ {} \arrow[rr, "\varphi_{12}"] & & {} \end{tikzcd}$$

$$\begin{array}{ccc} \varphi_1(U_1\cap U_2) & \overset{\varphi_{12}}{\dashrightarrow} & \varphi_2(U_1\cap U_2) \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^n \end{array}$$

$$\varphi_{12} : \varphi_1(U_1\cap U_2) \longrightarrow \varphi_2(U_1\cap U_2)$$

$\varphi_{12} = \varphi_2 \circ \varphi_1^{-1}$ restricted to $\varphi_1(U_1\cap U_2)$.

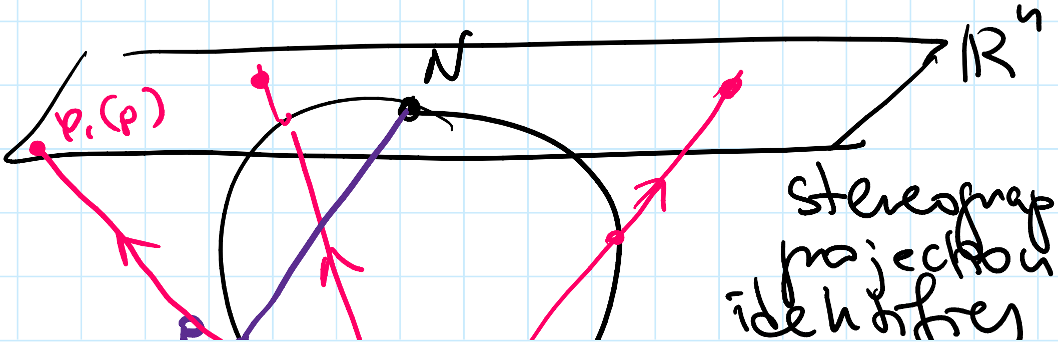
Clearly, φ_{12} is a homeomorphism.

<u>Note:</u> φ_{12} is a "gluing data" for U_1 and U_2, that is,

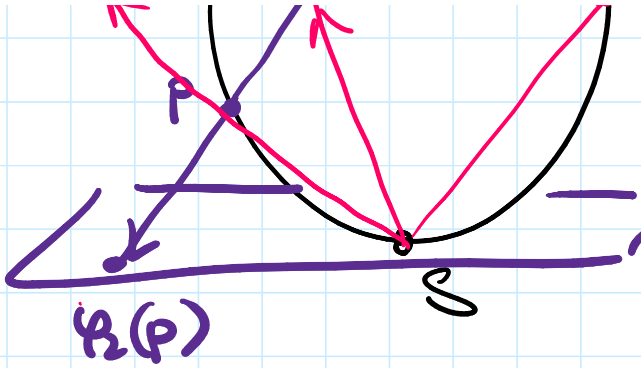
$U_1 \cup U_2 \cong \mathbb{R}^n$ glued with $\mathbb{R}^n$ along the identification

$$\begin{array}{ccc} \varphi_1(U_1\cap U_2) & \xrightarrow[\sim]{\varphi_{12}} & \varphi_2(U_1\cap U_2) \\ \| & & \| \\ \text{open subset in } \mathbb{R}^n & & \text{open subset in } \mathbb{R}^n \end{array}$$

<u>Ex:</u> $S^n = U_1 \cup U_2$



stereographic projection identifies



projection identifies points on top $\mathbb{R}^n$ with $S^n \setminus \{S\}$

$U_1 \simeq \mathbb{R}^n \qquad U_2 \simeq \mathbb{R}^n$

$$\underset{S^n \setminus \{N,S\}}{\overset{\parallel}{U_1 \cap U_2}} \simeq \mathbb{R}^n \setminus \{(0, \ldots, 0)\}$$

$S^n = \mathbb{R}^n \cup \mathbb{R}^n$ glued along the homeomorphism

$$\mathbb{R}^n \setminus \{(0..0)\} \overset{\varphi_{12}}{\simeq} \mathbb{R}^n \setminus \{(0, \ldots 0)\}$$

φ_{12} sends $\varphi_1(p)$ to $\varphi_2(p)$

Very interesting exercise to compute it explicitly! (related to inversion).

If we have more charts, it is the same: $U_1 \ldots U_N$

$\varphi_i : U_i \longrightarrow \mathbb{R}^n$

$\varphi_i : U_i \longrightarrow \mathbb{R}^n$

$\varphi_{ij} : \varphi_i(U_i \cap U_j) \xrightarrow{\sim} \varphi_j(U_i \cap U_j)$

gluing homeomorphisms.

__Def__ A __smooth__ (or differentiable) n-dimensional manifold is a topological manifold with charts as above $\varphi_i : U_i \to \mathbb{R}^n$ such that φ_{ij} are __smooth__ functions (C^∞ differentiable)

__Note:__ φ_{ij} is a function from an open subset in $\mathbb{R}^n_{x_1 \dots x_n}$ to another open subset in $\mathbb{R}^n_{y_1 \dots y_n}$, so this defn. makes sense:

$$(y_1 \dots y_n) = (\varphi_{ij}^{(1)}(x_1 \dots x_n), \varphi_{ij}^{(2)}(x_1 \dots x_n),$$

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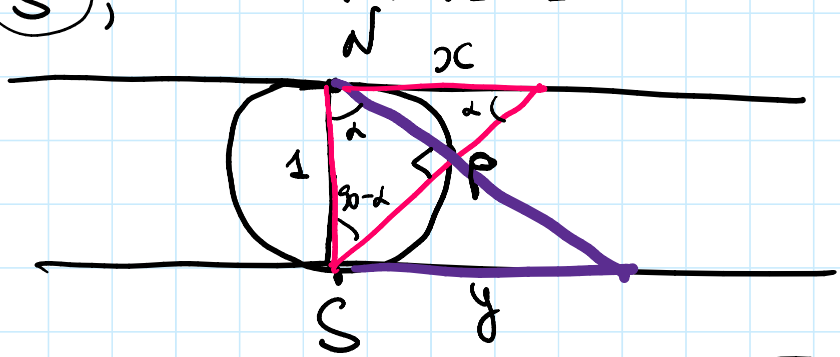
$\dots \varphi_{ij}^{(n)}(x_1 \dots x_n))$

all these are smooth functions.

Ex: S^1, diameter is 1



$$\tan\alpha = \frac{1}{x} = \frac{y}{1} \Rightarrow y = \frac{1}{x}$$

from red triangle

from blue triangle.

In the chart $S^1 \setminus \{S\}$ we identify the point p with the point $x \in \mathbb{R}^1$

In the chart $S^1 \setminus \{N\}$ we identify p with the point $y \in \mathbb{R}^1$

φ_{12} is defined when $x \neq 0$ and $y \neq 0$

and

$\varphi_{12}(x) = y = \frac{1}{\dots}$

and

$$\varphi_{12}(x) = y = \frac{1}{x}$$

Clearly, this is a C^∞ differentiable function of x.

$\Rightarrow S^1$ is a smooth manifold.

Similarly, S^n is a smooth manifold.

$\varphi_i : U_i \longrightarrow \mathbb{R}^n$ defines <u>local coordinates</u> on U_i

φ_{ij} = change of coordinates between φ_i and φ_j on $U_i \cap U_j$] transition function.

<u>Ex</u> $S^n, \mathbb{R}P^n, \mathbb{C}P^n, T^n, \ldots$ all smooth manifolds.

<u>Warning:</u> Not every top. manifold is smooth or it can be given a structure of a smooth manifold in many different ways!

in many different ways!

- There are examples of top. manifolds with NO smooth structure.

- Thm (Milnor) There are 28 different smooth structures on S^7!

Thm (Freedman) There are infinitely many smooth structures on $\mathbb{R}^4$!

Good news: in dim 2 and 3 everything is fine and any top. manifold has a unique smooth structure.

PL (piecewise linear) manifold:

X = simplicial complex

x = vertex of X

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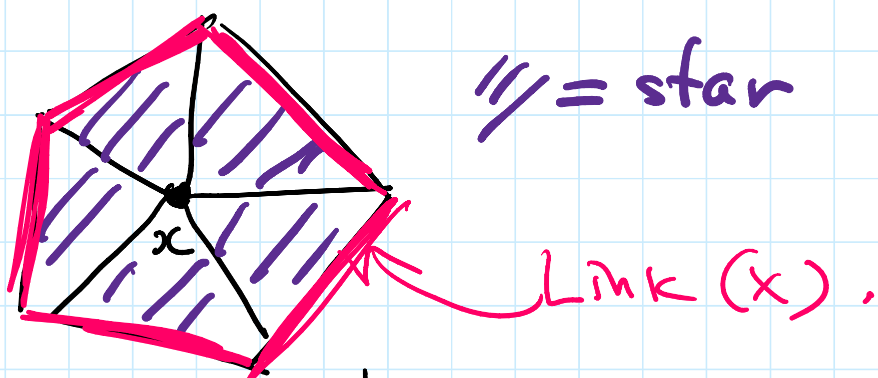
Star(x) = union of all simplices containing x

Link(x) = union of all faces of simplices in Star(x) not containing x.



Def A n-dim PL manifold is a simplicial complex such that the link of every vertex is a sphere S^{n-1}

Fact Top. manifolds $\supsetneq$ PL manifolds $\supsetneq$ Smooth manifolds

In general, all inclusions are strict

In dim=1, 2 and 3, all 3 classes are the same.

are the same.

<u>Warning</u> Triangulation Conjecture is false,

that is, there exists top. manifolds without triangulation.