Lecture 8 (4/14)

Wednesday, April 14, 2021 2:11 PM

Top. manifolds $\underset{\neq}{\supset}$ PL manifolds $\underset{\neq}{\supset}$ Smooth manifolds

(each point has a neighborhood homeomorphic to $\mathbb{R}^n$)

(φ_{ij} are differentiable)

Fact: Every compact, connected 1-manifold is homeomorphic to a circle S^1.

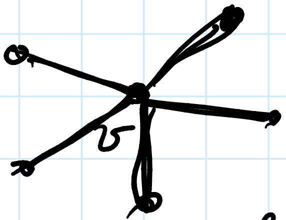
(compact, not connected $\Rightarrow$ disjoint union of circles).

Moreover, in dimension 1

(Top. manifolds) = (PL manifolds)
= (smooth manifolds).

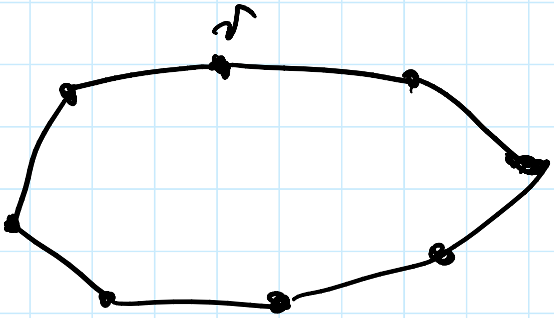
Note: PL 1-manifold is a simplicial complex built out of 1-simplices = segments of 1-simplices = segments

Link of every vertex is homeomorph. to a sphere



In this case
Link(v) = other ends of segments at v

$= S^0 \Rightarrow$ exactly 2 segments at each vertex.



because our 1-manifold is compact, this should close to a cycle.

Surface = 2-manifold.

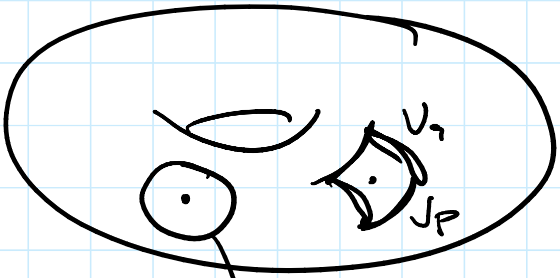
Fact: (harder) For compact surfaces, (top. surface) = (PL surface) = (smooth surface)

Any compact top. surface has a unique

any top-surface has a unique smooth structure.

Examples of surfaces:

① S^2, $\mathbb{R}P^2$, $T^2 = \underline{\underline{S^1}} \times \underline{\underline{S^1}}$



every point on a torus has a neighborhood homeomorphic to an open ball in $\mathbb{R}^2 \simeq \mathbb{R}^2$

Another proof: a point p in S^1 has a neighborhood U_p homeomorphic to $\mathbb{R}$

$(p,q) \in S^1 \times S^1 = T^2$

$$\underset{\mathbb{R}}{\underset{\wr\!\!\wr}{U_p}} \times \underset{\mathbb{R}}{\underset{\wr\!\!\wr}{U_q}} \subset S^1 \times S^1 = T^2$$

$$U_p \times U_q \simeq \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

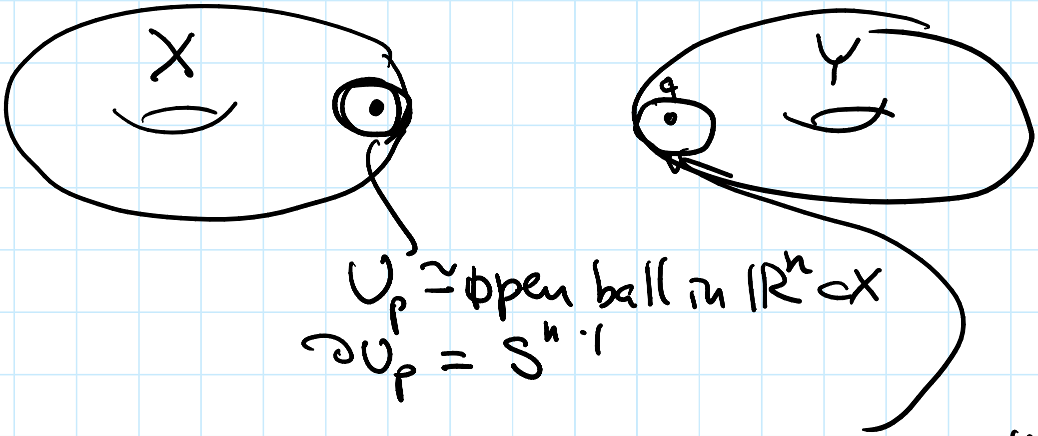
Rmk X, Y = smooth manifolds

$\Rightarrow X \times Y$ is a smooth manifold.

Connected sum

Suppose that X, Y are top. manifolds of same dimension n

Pick a point $p \in X$, $q \in Y$

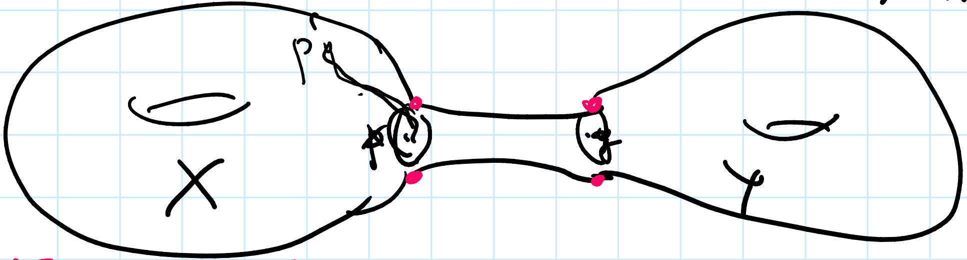


$U_p \simeq$ open ball in $\mathbb{R}^n \subset X$
$\partial U_p = S^{n-1}$

$U_q =$ open ball in Y
$\partial U_q = S^{n-1}$

$X \# Y =$ result of gluing of $(X \setminus U_p)$ and $(Y \setminus U_q)$ along this S^{n-1}

$X \# Y$

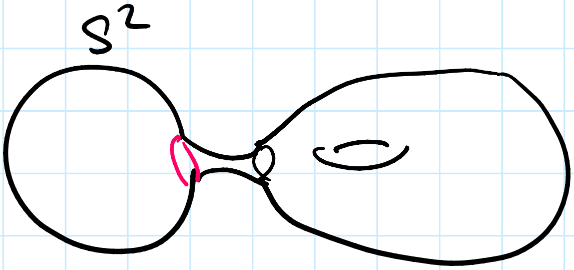


Easy fact: we can smooth out the corners and make $X \# Y$

a manifold.

This works in any dimension, and does not depends on the choice of points p and q provided that X and Y are connected.

Ex: $T^2 \# S^2 \simeq T^2$



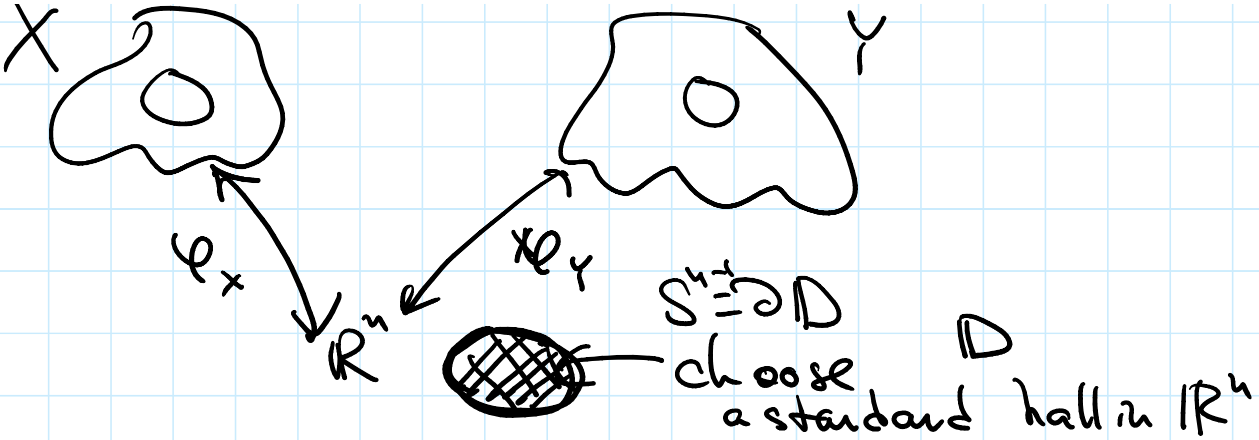
In fact, $S^2 \# M = M$ for any surface M

$S^n \# M = M$ for any n-manifold M,

with the standard smooth structure.

Proof: Complement to a disk in S^2 is again a disk, and $S^2 \# M =$

$= (\text{disk}) \cup (M \setminus \text{disk}) = M$. ∎



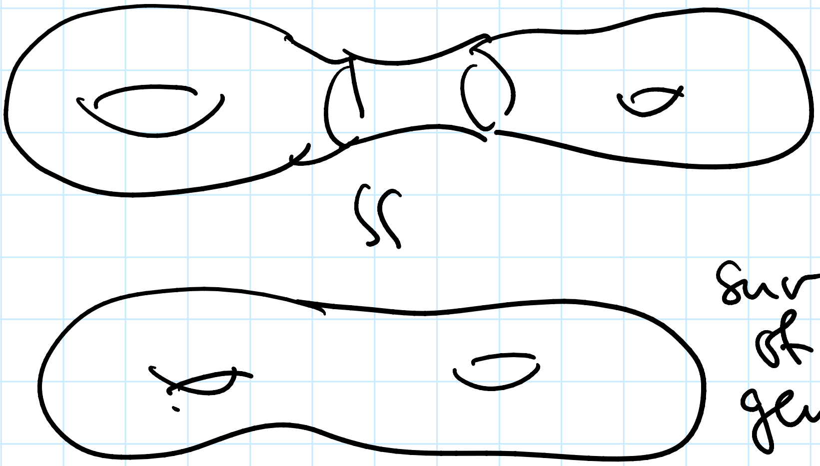
$$X \setminus (\varphi_X^{-1}(D)) \cup (Y \setminus (\varphi_Y^{-1}(D)))$$

Fact This does not depend on a choice of chart in X and Y for fixed structure of top/smooth manifold.

Rmk $\{$all different smooth structures on $S^n\}$ $= \{$exotic $S^n\}$ form a group under connect sum, and the standard S^n is a unit in this group.

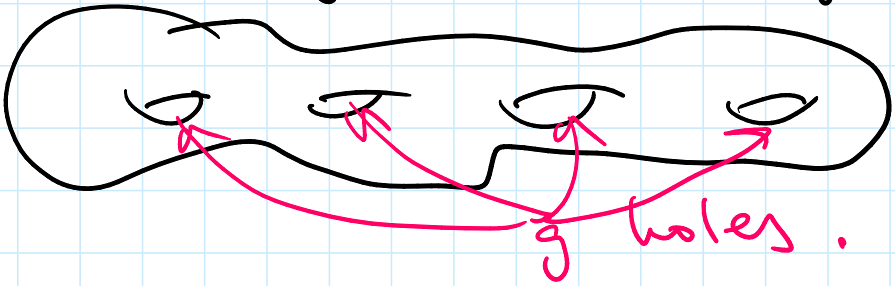
$\{$exotic $S^7, \#\} \simeq \mathbb{Z}_{28}$.

$T^2 \# T^2$



surface of genus 2

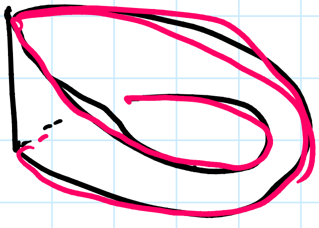
$$\underbrace{T^2 \# T^2 \# T^2 \# \_\_ \# T^2}_{g \text{ times}} = \text{surface of genus } g$$



Exercise $\mathbb{R}P^2 \setminus (\text{disk}) \simeq$ Mobius band.

$\Longleftarrow$ neighborhood of the "infinite line"

$\underset{\underset{S^1}{\shortparallel}}{\mathbb{R}P^1} \subset \mathbb{R}P^2$ is homeo to Mobius band



$\boxed{\mathbb{R}P^2 \# M}$ = first take out a

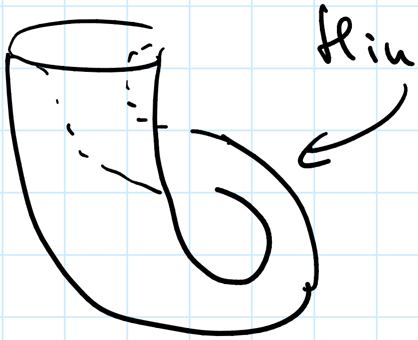
disk from M and

disk from M and then glue in Mobius band along the boundary,

(∂(Möbius band) $= S^1$)

Ex $\underbrace{\mathbb{RP}^2}\#\underbrace{\mathbb{RP}^2} = \left(\begin{matrix}\text{Mobius}\\ \text{band}\end{matrix}\right) \cup \left(\begin{matrix}\text{Mobius}\\ \text{band}\end{matrix}\right)$ along boundary.

Exercise: this is the Klein bottle,



Hint: find Mobius bands in this picture!

Exercise $\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2 = \mathbb{RP}^2 \#$ Klein bottle

$= \mathbb{RP}^2 \# T^2$

Thm(!) Any compact, connected (classification of surfaces) surface is either $\underbrace{T^2\#T^2\#\dots\#T^2}_{g\text{ times}}$

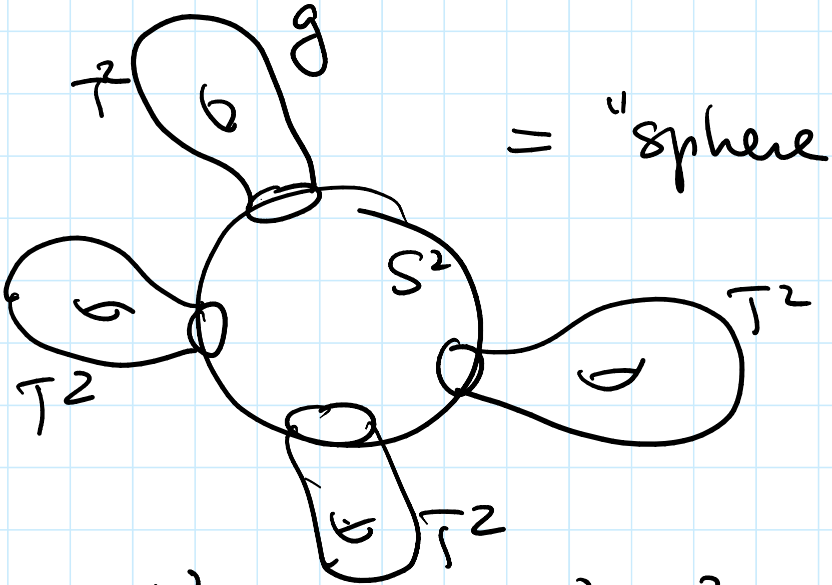
or $\underbrace{\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2\dots\#\mathbb{RP}^2}$

or $\underbrace{\mathbb{RP}\#\mathbb{RP}\#\mathbb{RP}\ldots\#\mathbb{RP}}_{k \text{ times}}$.

(b) Furthermore, all these are pairwise not homeomorphic.

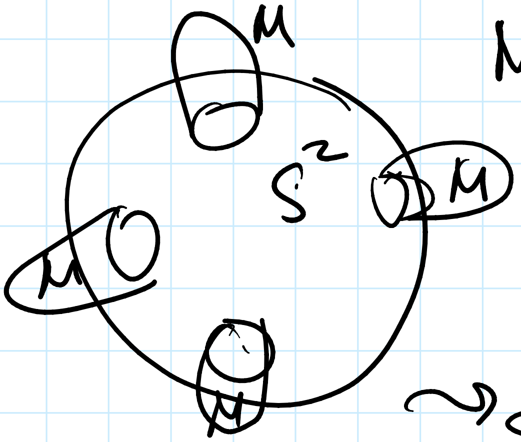
Another way to visualize these:

$$\underbrace{T^2\#\ldots\#T^2}_{g} = S^2\#(\underbrace{T^2\#\ldots\#T^2}_{g})$$



= "sphere with g handles"

$$\underbrace{\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2}_{k} = S^2\#(\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2)$$



M = Mobius band attached to a circle in S^2 along the boundary.

$\rightsquigarrow$ S^2 with k Mobius bands

$\rightsquigarrow S^2$ with k Mobius bands attached.