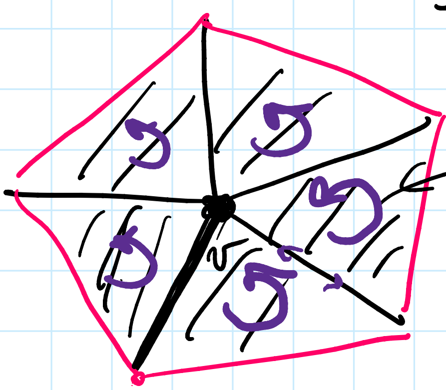
Orientation for surfaces

M = surface (connected, compact)

As discussed last time, can triangulate M and make it a PL manifold



Star(v) = all simplices containing v

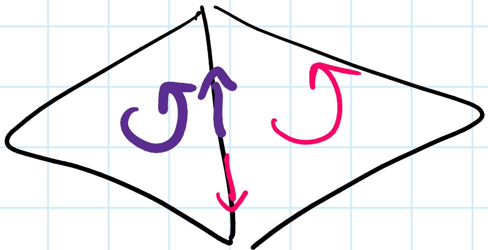
local picture at v

Link(v) = all faces of Star(v) not containing v

$\text{Link}(v) \cong S^1$

every edge bounds exactly 2 triangles

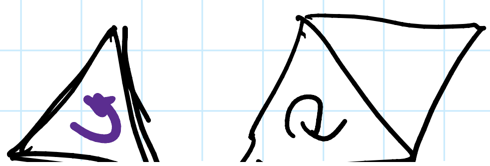
Def An orientation on M (as a PL manifold) is a choice of orientation on all triangles in a triangulation such that for each edge these induce for each edge these induce opposite orientation on the edge.



M is called orientable if it has an orientation, and non-orientable otherwise.

<u>Fact</u> If M is orientable and connected, it has exactly 2 orientations, and M orientable with r connected components, it has 2^r orientations.

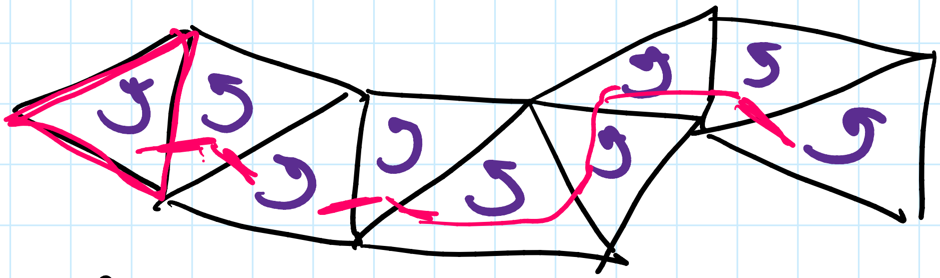
<u>Proof</u>: 1) Assume M is connected. Pick a simplex, it has two possible orientations. Once we chose



one of these,

 one of these,

this determines orientation of all neighboring simplices, ~~and~~ so on. Since M is connected, any compact
2 simplices are connected by a chain



If M is orientable, there are no contradictions $\Rightarrow$ uniquely extend orientation to all triangles.

Change orientation of the initial simplex $\Rightarrow$ change orientation for all others.

2) If M is disconnected, choose orientation for each component separately. ∎

Facts (a) S^2, T^2 are orientable

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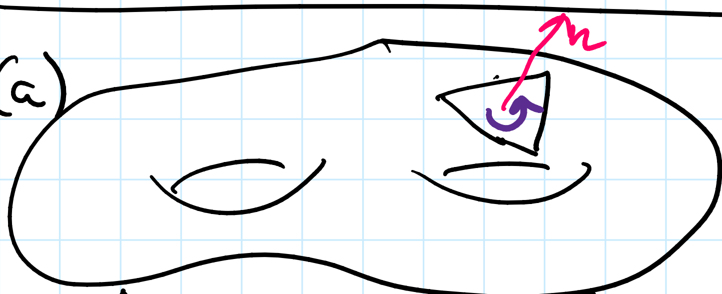
(b) X and Y orientable $\Rightarrow$ $X \# Y$ orientable

(c) $\Rightarrow$ $T^2 \# T^2 \# \ldots \# T^2$ = genus g surface is orientable

(d) $\mathbb{RP}^2$ is not orientable, as well as $\mathbb{RP}^2 \# Y$ for any Y.

(a)

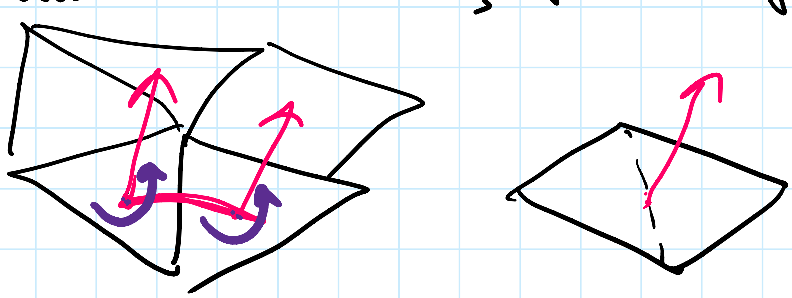


$\mathbb{R}^3$ has 1 orientation.

M = closed surface in $\mathbb{R}^3$ (for example, S^2 or T^2)

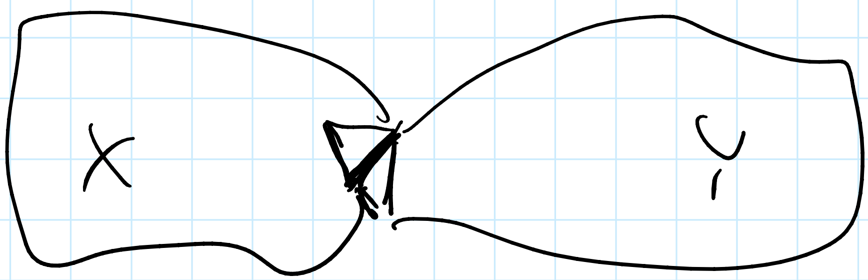
At every simplex, choose a normal vector which points outside

Orient this simplex right hand rule: counter-clockwise, if look from the end of $\underline{n}$.

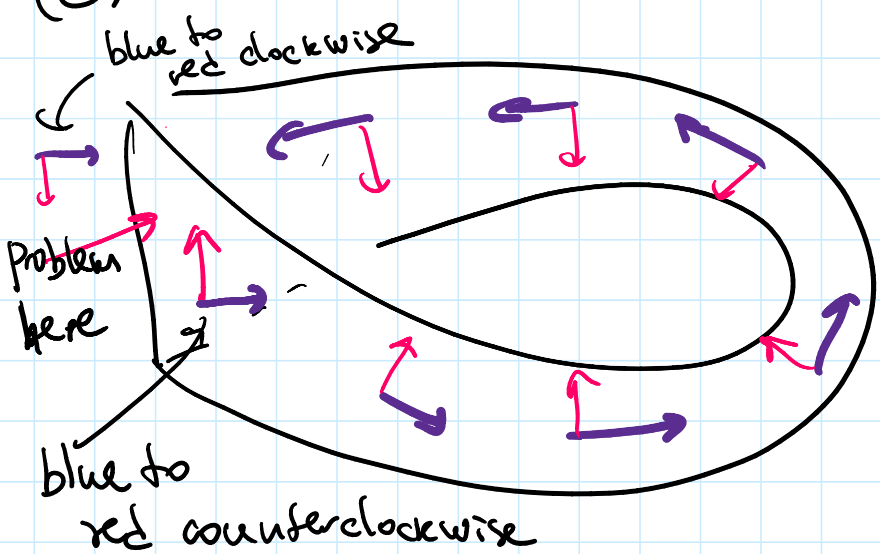


This is a correct orientation $\Rightarrow M$ is orientable.

(b) We can realize $X \# Y$ by removing one simplex from X and one from Y, choosing opposite orientations on these & extending to orientations at X and Y.



(d) Möbius band is not orientable!



$\mathbb{R}P^2 = (\text{Disk}) \cup (\text{Mobius band})$

$\mathbb{RP}^2 = (\text{Disk}) \cup (\underline{\text{Mobius band}})$

contains non-orientable piece

$\Rightarrow$ not orientable!

$\mathbb{RP}^2 \# Y = (Y \setminus \text{disk}) \cup (\text{Mobius band})$

$\Rightarrow$ not orientable.

What do we know about homology & cohomology of surfaces?

<u>Thm</u> (a) If M is orientable surface, connected then $H_2(M) = \mathbb{Z}$

$$H^2(M) = \mathbb{Z}$$

(b) If M is non-orientable connected surface, then

$H_2(M) = 0 \quad H^2(M) = \mathbb{Z}_2$.

<u>Cor</u> $\text{Tors}(H^2) = \text{Tors}(H_1)$

$\Rightarrow$ if M orientable, H_1 has no torsion

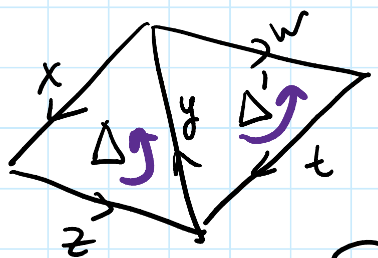
if M non-orientable $\text{Tors}(H_1) \cong \mathbb{Z}_2$

if M non-orientable, $\mathrm{Tors}(H_1) \simeq \mathbb{Z}_2$.

<u>Proof</u> We want to compute $H_2(M)$ with various coefficients ($\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Z}_p$ $p>2$).

(a) $0 \longrightarrow C_2(M) \xrightarrow{\partial} C_1(M)$

$$H_2(M) = \mathrm{Ker}\,\partial \quad (\text{no } C_3).$$



With $\mathbb{Z}$ coefs:

$$\partial(\Delta) = x+y+z$$

$$\partial(\Delta') = -y-w-t$$

Suppose that we have a class in

$$C_2 \ni \alpha = a\cdot\Delta + b\cdot\Delta' + \dots$$

$$\partial\alpha = a\cdot y + b(-y) + \dots$$

just pick coef. at y.

This cancels out if $\underline{a=b}$
if Δ and Δ' have same orientation

or $a = -b$ if Δ and Δ'

or $a=-b$ if Δ and Δ'

have opposite orientations.

2) if M is orientable, we can just

pick $\alpha =$ (sum of all 2-simplices with compatible orientations)

$\Rightarrow \partial \alpha = 0$ and $\operatorname{Ker}\partial$ is spanned by α.

If we pick a coefficient at one simplex, it determines coefficients at all other simplices.

$\Rightarrow \operatorname{Ker}\partial \simeq \mathbb{Z} = H_2(M)$

The same proof works for any coefs.

By universal coefficient theorem, this implies $H^2(M)=\mathbb{Z}$.

Rmk if H^2 has p-torsion, then

$H^2(M,\mathbb{Z}_p)$ is bigger

but $\dim H^2(M,\mathbb{Z}_p) = \dim H_2(M,\mathbb{Z}_p) = 1$.

Cochain complex splits as a direct sum of $\mathbb{Z}$, $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$

$$\to C^1 \to C^2 \to 0$$

$\mathbb{Z}$ ← only one

$\underline{\mathbb{Z} \xrightarrow{1} \mathbb{Z},}$

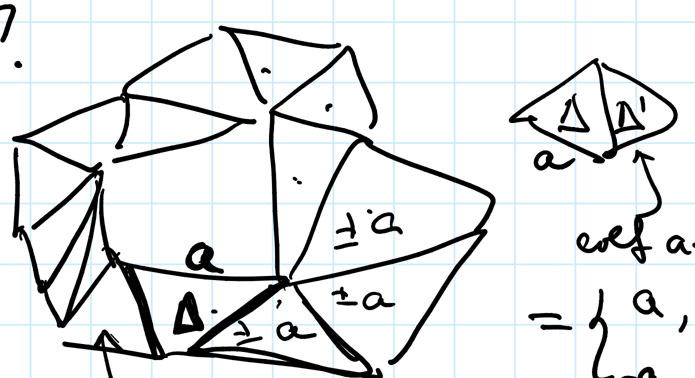
$\mathbb{Z} \xrightarrow{n} \mathbb{Z} \quad n > 1$

Choose p prime dividing n

$\Rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \cong \mathbb{Z} \xrightarrow{0} \mathbb{Z}$ mod p

$\Rightarrow H^2$ is bigger!

(b) What's different in non-orientable case?



coef at Δ' $= \begin{cases} a, \text{ if same orient} \\ -a, \text{ opposite} \end{cases}$

$-a$ all coefs at 2-simplices are $\pm a$ depending on orientation

If we change orientation,

If we change orientation,

we get $a = -a$

Over $\mathbb{Z}$ or over $\mathbb{Z}_p$, $p>2$

$\Rightarrow \alpha = 0$ and $H_2(M,\mathbb{Z}) = 0$
$H_2(M,\mathbb{Z}_p) = 0$

Over $\mathbb{Z}_2$, this is actually fine!

We can α = sum of all 2-simplices with coefficient 1!

$\partial \alpha = \left(\begin{array}{l} \text{some edges with coef } 0 \\ \text{some edges with coef } \underset{0 \bmod 2}{\overset{\shortparallel}{2}} \end{array} \right)$

$\Rightarrow H_2(M,\mathbb{Z}_2) = \mathbb{Z}_2$

By universal coefficient theorem,

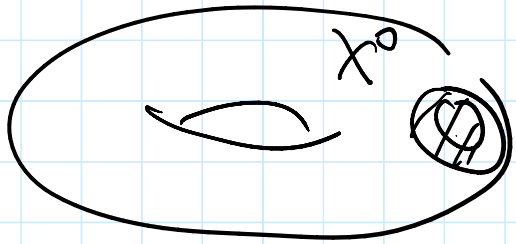
this implies $H^2(M,\mathbb{Z}_p) = 0 \quad p>2$

$H^2(M,\mathbb{Z}_2) = \mathbb{Z}_2$

$\Rightarrow H^2(M,\mathbb{Z}) = \mathbb{Z}_2$

(by same argument with

(by same argument with splitting to 2-term complexes).



$X = X^0 \cup D^2$

$X^0 \cap D^2 \simeq S^1$

$\Rightarrow$ use Mayer-Vietoris

$$\to H_i(S^1) \to H_i(X^0) \oplus H_i(D^2) \to H_i(X) \to$$
