MAT 21A, solutions to practice problems for the final exam

1. Compute the limit:

a) $\lim_{x\to 3} e^{1/x}$

Answer: $e^{1/3}$.

b) $\lim_{x\to 3} \ln(x-3)$

Answer: $-\infty$.

c) $\lim_{x\to 3} \frac{\ln(x-2)}{x-3}$

Solution: As x approaches 3, $(x-2)$ approaches 1, so $\ln(x-2)$ approaches $ln(1) = 0$. Therefore we have a limit of the form $0/0$ and can apply the L'Hôpital's rule:

$$\lim_{x\to 3} \frac{\ln(x-2)}{x-3} = \lim_{x\to 3} \frac{1/(x-2)}{1} = 1.$$

d) $\lim_{x\to\infty} \frac{8x^5-7x^3+9}{(3x^2-1)(2x^3-3)}$

Solution: Let us divide the top and the bottom of this fraction by x^5:

$$\lim_{x\to\infty} \frac{8x^5-7x^3+9}{(3x^2-1)(2x^3-3)} = \lim_{x\to\infty} \frac{8-7/x^2+9/x^5}{(3-1/x^2)(2-3/x^3)} = \frac{8-0+0}{(3-0)(2-0)} = \frac{4}{3}.$$

e) $\lim_{x\to\infty} \frac{e^x}{x^3}$

Solution: As this is the limit of the type ∞/∞, we can apply the L'Hôpital's rule several times:

$$\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} \frac{e^x}{3x^2} = \lim_{x\to\infty} \frac{e^x}{6x} = \lim_{x\to\infty} \frac{e^x}{6} = \infty.$$

f) $\lim_{x\to 0} \frac{\arctan(x)-x}{x^3}$

Solution: As this is the limit of the type $0/0$, we can apply the L'Hôpital's rule several times:

$$\lim_{x\to 0} \frac{\arctan(x)-x}{x^3} = \lim_{x\to 0} \frac{1/(1+x^2)-1}{3x^2} = \lim_{x\to 0} \frac{1-1-x^2}{3x^2(1+x^2)} =$$

$$\lim_{x\to 0} \frac{-x^2}{3x^2(1+x^2)} = \lim_{x\to 0} \frac{-1}{3(1+x^2)} = \frac{-1}{3}.$$

2. Compute the derivative of the following functions:

a) $f(x) = x \ln x - x$

Answer: $\ln(x)$

b) $f(x) = e^{3x^2}$

Answer: $6xe^{3x^2}$.

c) $f(x) = (x-1)^5 \cos x$

Answer: $5(x-1)^4 \cos x - (x-1)^5 \sin x$.

d) $f(x) = \sin(\ln x)$

Answer: $\frac{\cos(\ln x)}{x}$.

e) $f(x) = \frac{e^{3x+2}}{\cos x+2}$

Answer: $\frac{(3\cos x+6+\sin x)e^{3x+2}}{(\cos x+2)^2}$.

3. Find the minimal and maximal values of a function:

a) $f(x) = x^2 e^{-x}$ on $[0,1]$

Solution: We have $f'(x) = 2xe^{-x} - x^2 e^{-x} = x(2-x)e^{-x}$, so $f'(x) = 0$ for $x = 0$ and $x = 2$. Since $x = 2$ is not on the interval and $x = 0$ is one of the endpoints, we just need to compute $f(0) = 0$ with $f(1) = e^{-1}$. Therefore the minimal value is 0 and the maximal value is e^{-1}.

b) $f(x) = 2x^3 - 3x^2 + 1$ on $[-1,2]$

Solution: We have $f'(x) = 6x^2 - 6x = 6x(x-1)$, so $f'(x) = 0$ for $x = 0$ and $x = 1$. We need to compute

$$f(-1) = -2-3+1 = -4,\ f(0) = 1,\ f(1) = 2-3+1 = 0,\ f(2) = 16-12+1 = 5.$$

Therefore the minimal value is (-4) and the maximal value is 5.

c) $f(x) = \sin^2 x$ on $[0,\pi]$

Solution: We have $f'(x) = 2\sin x \cos x$, so $f'(x) = 0$ when either $\sin x = 0$ (so $x = \pi k$) or $\cos x = 0$ (so $x = \pi/2 + \pi k$). On the interval $[0,\pi]$ we have 3 critical numbers $0, \pi/2, \pi$ and both endpoints are among them. Therefore one needs to compute $f(0) = 0,\ f(\pi/2) = 1,\ f(\pi) = 0$, and the minimal and maximal values are 0 and 1 respectively.

d) $\frac{x}{1+x^2}$ on $[-2,2]$

Solution: We have $f'(x) = \frac{1+x^2-x(2x)}{(1+x^2)^2} = 1 - x^2(1+x^2)^2$, and the critical numbers are $x = 1$ and $x = -1$. We have

$$f(-2) = -2/5,\ f(-1) = -1/2,\ f(1) = 1/2,\ f(2) = 2/5.$$

Since $1/2 > 2/5$, the minimal and maximal values are $-1/2$ and $1/2$ respectively.

4. Find the equation of the tangent line to the graph of $f(x) = \ln x$ at $x = 5$.

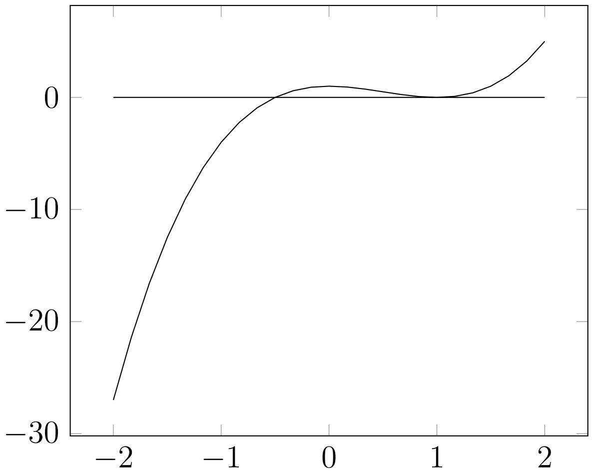
Answer: $y = \frac{1}{5}x + (\ln 5 - 1)$.

5. For a given function:

- Find the domain
- Determine the equations of vertical and horizontal asymptotes
- Find the derivative and determine the intervals where the function is increasing/decreasing
- Find the second derivative and determine the intervals where the function is concave up/down, find inflection points
- Draw the graph using all the information above

a) $f(x) = 2x^3 - 3x^2 + 1$

Solution: The function is defined everywhere, and there are no asymptotes. We have $f'(x) = 6x^2 - 6x = 6x(x-1)$, so the function is increasing for $x > 1$ and $x < 0$, and decreasing for $0 < x < 1$. Furthermore, $f''(x) = 12x - 6$, so the function is concave up for $x > 1/2$ and concave down for $x < 1/2$, and it has an inflection point at $x = 1/2$.



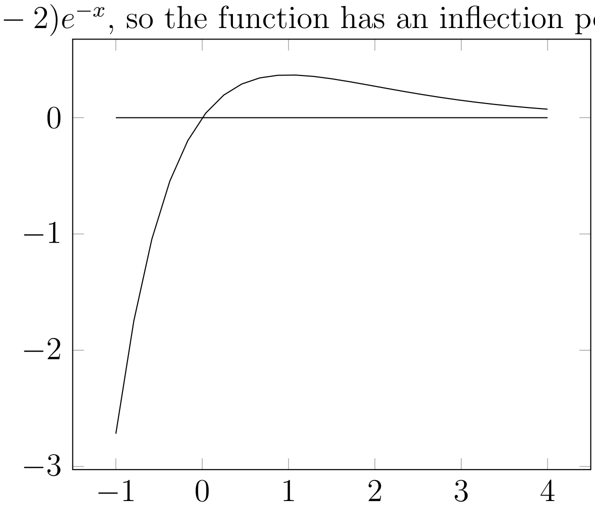
b) $f(x) = xe^{-x}$

Solution: The function is defined everywhere. To find the asymptotes, remark that at $x \to -\infty$ the function e^{-x} goes to infinity, so $\lim_{x\to-\infty} xe^{-x} = -\infty$. To find the limit at $x \to +\infty$, let us use the L'Hôpital's rule:

$$\lim_{x\to+\infty} xe^{-x} = \lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0.$$

Therefore the graph has a horizontal asymptote $y = 0$ at $x \to +\infty$.

Now $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$, so the function increases for $x < 1$ and decreases for $x > 1$. Furthermore, $f''(x) = -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}$, so the function has an inflection point at $x = 2$.



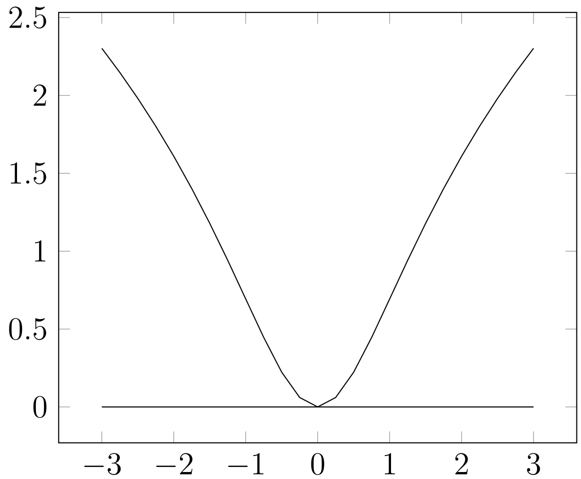
c) $f(x) = \ln(x^2 + 1)$

Solution: Since $x^2+1 \geq 1$, the function is defined and nonnegative everywhere. As x approaches $\pm\infty$, $x^2+1 \to +\infty$, so $\ln(x^2+1) \to +\infty$, and there are no horizontal asymptotes.

Now $f'(x) = \frac{2x}{x^2+1}$, so the function decreases for $x < 0$ and increases for $x > 0$. Furthermore,

$$f''(x) = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \frac{2(1-x^2)}{(x^2+1)^2},$$

and the function has inflection points at $x = \pm 1$.

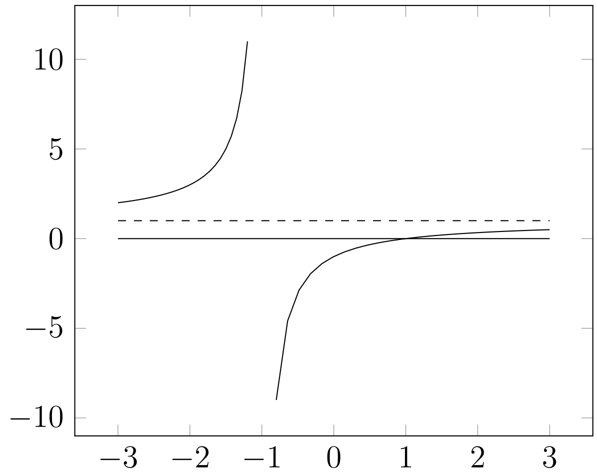


d) $f(x) = \frac{x-1}{x+1}$

Solution: The function is defined for $x \neq -1$, and has a vertical asymptote at $x = -1$. To find horizontal asymptotes, we have

$$\lim_{x\to\infty} \frac{x-1}{x+1} = \lim_{x\to\infty} \frac{1-1/x}{1+1/x} = 1.$$

Now $f'(x) = \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$, so the function increases on every interval where it is defined. Furtermore, $f''(x) = -4(x+1)^3$, so the function is concave down for $x > -1$ and concave up for $x < -1$.



6. Consider the function

$$f(x) = \begin{cases} x+1, & \text{if } x < -1 \\ x^2 + ax + b, & \text{if } x \geq -1. \end{cases}$$

a) For which values of the parameters it is continuous?

Solution: The function is continuous, if its limits from the left and from the right at $x = -1$ coincide:

$$(-1) + 1 = (-1)^2 + a(-1) + b \;\Leftrightarrow\; 0 = 1 - a + b \;\Leftrightarrow\; b = a - 1.$$

b) For which values of the parameters it has a derivative at every point?

Solution: The function clearly has a derivative for all $x \neq -1$, and it has derivative at $x = -1$ if it is continuous at this point (see (a)), and the derivatives from the left and from the right coincide: $1 = 2(-1) + a$, so $a = 3$ and $b = 2$.

7. Consider the curve given by the equation $x^{2/3} + y^{2/3} = 1$. Find y' using implicit differentiation and sketch this curve. (Note: assume that $x^{2/3} = \sqrt[3]{x^2}$ is defined for all x).

Answer: $y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$.

8. Consider the curve given by the equation $y^2 = x^3 - x$.
a) Find y' using implicit differentiation.

b) Find the equation of the tangent line at the point $(2, \sqrt{6})$.

Answer: a) $y' = \frac{3x^2-1}{2y}$.
b) $y = \frac{11}{2\sqrt{6}}(x-2) + \sqrt{6}$.

9. An open rectangular box with square base is to be made from 1 area unit of material. What dimensions will result in a box with the largest possible volume ?

Solution: Let h be the height of the box and let x be the size of the base. Then the surface area equals $x^2 + 4xh = 1$, so

$$h = (1 - x^2)/4x.$$

The volume then equals $V(x) = x^2h = x(1-x^2)/4 = (x - x^3)/4$. We get $V'(x) = (1 - 3x^2)/4$, so the maximal volume is at

$$x = \frac{1}{\sqrt{3}},\ h = \frac{1 - 1/3}{4/\sqrt{3}} = \frac{2\sqrt{3}}{12} = \frac{1}{2\sqrt{3}}.$$

10. A TV set costs \$100. If its price is lowered by $a\%$, the sales would increase by $2a\%$. Find the discount amount a which yields the maximal profit.

Answer: 25%