MAT 21A, Fall 2021
Solutions to homework 3

1. (10 points) Find the limit $\lim_{x\to 1} \arccos(\ln(\sqrt{x}))$.

Solution: We have $\lim_{x\to 1} \sqrt{x} = 1$, so $\lim_{x\to 1} \ln(\sqrt{x}) = \ln(1) = 0$ and

$$\lim_{x\to 1} \arccos(\ln(\sqrt{x})) = \arccos(0) = \frac{\pi}{2}.$$

2. (10 points) Show that the function $F(x) = (x-1)^2(x-5)^2 + x$ takes on value 3 for some value of x.

Solution: We have $F(1) = (1-1)^2(1-5)^2 + 1 = 0 + 1 = 1, F(5) = (5-1)^2(5-5)^2 + 5 = 0 + 5 = 5$. Therefore $F(1) < 3, F(5) > 3$ and by Intermediate Value Theorem $F(x) = 3$ for some point x on the interval $[1, 5]$.

3. (10 points) Compute the limit

$$\lim_{x\to +\infty} \frac{3x^3 + 2x^2 - 5x + 1}{-2x^3 + x^2 - 4x + 7}.$$

Solution: We divide the numerator and the denominator by the highest power of x in the fraction, that is, by x^3:

$$\lim_{x\to +\infty} \frac{3x^3 + 2x^2 - 5x + 1}{-2x^3 + x^2 - 4x + 7} = \lim_{x\to +\infty} \frac{3 + 2/x - 5/x^2 + 1/x^3}{-2 + 1/x - 4/x^2 + 7/x^3} = \frac{3+0+0+0}{-2+0+0+0} = -\frac{3}{2}.$$

4. (10 points) Consider the function $f(x) = \frac{3}{2}\left(\frac{x}{x-1}\right)^{\frac{3}{4}}$.

a) Find the domain of $f(x)$.
b) How does the graph behave as $x \to 0$?
c) How does the graph behave as $x \to +1$ and $x \to -1$?
d) Find the vertical and horizontal asymptotes for $f(x)$.
e) Sketch the graph of $f(x)$.

Solution: (a) The function is defined where $x \neq 1$, and $\frac{x}{x-1} \geq 0$. For $x > 1$ we have $x > 0, x - 1 > 0$, so $\frac{x}{x-1} > 0$; for $0 < x < 1$ we have $x > 0, x - 1 < 0$, so $\frac{x}{x-1} >< 0$; for $x < 0$ we have $x < 0, x - 1 < 0$, so $\frac{x}{x-1} > 0$. Therefore the domain is $(-\infty, 0] \cup (1, +\infty)$.

(b) As $x \to 0^-$, we have $x - 1 \to -1$, so $\frac{x}{x-1} \to 0^+$. Therefore $\left(\frac{x}{x-1}\right)^{\frac{3}{4}} \to 0^{\frac{3}{4}} = 0$, and $\lim_{x\to 0^-} f(x) = 0$. The function is continuous at $x = 0$.

(c) For $x \to 1^+$ we have $x - 1 \to 0$, so $\frac{x}{x-1} \to +\infty$. Therefore $\left(\frac{x}{x-1}\right)^{\frac{3}{4}} \to +\infty$, and $\lim_{x\to 1^+} f(x) = +\infty$.

For $x \to -1$ we have

$$\lim_{x\to -1} f(x) = \frac{3}{2}\left(\frac{-1}{-2}\right)^{\frac{3}{4}} = \frac{3}{2\cdot 2^{3/4}}.$$

(d) Since $\lim_{x\to 0^-} f(x) = 0$ and $\lim_{x\to 1^+} f(x) = +\infty$, the function $f(x)$ has no vertical asymptote at $x = 0$ but it has a vertical asymptote at $x = 1$. Furthermore, we have

$$\lim_{x\to\pm\infty} \frac{x}{x-1} = \lim_{x\to\pm\infty} \frac{1}{1-1/x} = 1,$$

Therefore at $x \to \pm\infty$ we have $\left(\frac{x}{x-1}\right)^{\frac{3}{4}} \to 1^{3/4} = 1$ and $\lim_{x\to\pm\infty} f(x) = \frac{3}{2}$. The function has a horizontal aymptote at $y = \frac{3}{2}$.

(e)