MAT 21A, Fall 2021
Solutions to homework 7

Find the limits using L'Hôpital's Rule:

1. $\lim_{x\to 1} \frac{\ln(x)}{x-1}$

Solution: We have

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{(\ln(x))'}{(x-1)'} = \lim_{x\to 1} \frac{1/x}{1} = 1.$$

2. $\lim_{x\to 0} \frac{e^x+e^{-x}-2}{x^2}$

Solution: We apply L'Hôpital's Rule twice:

$$\lim_{x\to 0} \frac{e^x+e^{-x}-2}{x^2} = \lim_{x\to 0} \frac{(e^x+e^{-x}-2)'}{(x^2)'} = \lim_{x\to 0} \frac{e^x-e^{-x}}{2x} =$$

$$\lim_{x\to 0} \frac{(e^x-e^{-x})'}{(2x)'} = \lim_{x\to 0} \frac{e^x+e^{-x}}{2} = \frac{e^0+e^0}{2} = \frac{1+1}{2} = 1.$$

3. $\lim_{x\to +\infty} \frac{e^{0.1x}}{x\sqrt{x}}$

Solution: We apply L'Hôpital's Rule twice:

$$\lim_{x\to +\infty} \frac{e^{0.1x}}{x\sqrt{x}} = \lim_{x\to +\infty} \frac{(e^{0.1x})'}{(x^{3/2})'} = \lim_{x\to +\infty} \frac{0.1e^{0.1x}}{3/2x^{1/2}} =$$

$$\lim_{x\to +\infty} \frac{(0.1e^{0.1x})'}{(3/2x^{1/2})'} = \lim_{x\to +\infty} \frac{0.1\cdot 0.1e^{0.1x}}{3/2\cdot 1/2x^{-1/2}}.$$

Now as $x \to +\infty$, we get $e^{0.1x} \to +\infty$ and $x^{-1/2} \to 0$, so

$$\lim_{x\to +\infty} \frac{0.1\cdot 0.1e^{0.1x}}{3/2\cdot 1/2x^{-1/2}} = +\infty.$$

4. For the function $f(x) = e^{-x^2}$:

(a) Find the domain, vertical and horizontal asymptotes
(b) Find the intervals where the function is increasing or decreasing
(c) Find the intervals wher the function is concave up or down
(d) Graph the function using this information

Solutions: (a) The function is defined for all x, so the domain is $(-\infty, +\infty)$. There are no vertical asymptotes.

For $x \to \pm\infty$ we have $x^2 \to +\infty$ and $-x^2 \to -\infty$, so

$$\lim_{x\to \pm\infty} e^{-x^2} = 0,$$

and the graph has horizontal asymptotes $y = 0$ at $x \to \pm\infty$.

(b) By Chain Rule we get $f'(x) = -2xe^{-x^2}$, so $f'(x) > 0$ if $-2x > 0$, so that $x < 0$. The function is increasing for $x < 0$ and decreasing for $x > 0$.

(c) By Product Rule we get

$f''(x) = -2e^{-x^2} - 2x(e^{-x^2})' = -2e^{-x^2} - 2x\cdot(-2x)e^{-x^2} = -2e^{-x^2} + 4x^2e^{-x^2} = (4x^2-2)e^{-x^2}$.

The function is concave up if $f''(x) > 0$, so that

$$4x^2 - 2 > 0, \quad 4x^2 > 2, \quad x^2 > \frac{1}{2},$$

so either $x > \frac{1}{\sqrt{2}}$ or $x < -\frac{1}{\sqrt{2}}$. The function is concave up on $(-\infty, -\frac{1}{\sqrt{2}}]$ and $[\frac{1}{\sqrt{2}}, +\infty)$ and concave down on $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$.

(d)