MAT 21B, Fall 2016
Solutions to Homework Assignment 4

Section 6.1: 62. (10 points) The arch $y = \sin x, 0 \le x \le \pi$ is revolved about the line $y = c, 0 \le c \le 1$ to generate a solid.

a) Find the value of c which minimizes the volume of the solid.

b) Find the value of c which maximizes the volume of the solid.

Solution: The volume of the solid is given by the equation:

$$V(c) = \int_0^\pi \pi(\sin x - c)^2 dx = \pi \int_0^\pi (\sin^2 x - 2c\sin x + c^2)dx.$$

Now

$$\int_0^\pi c^2 dx = \pi \cdot c^2, \ \int_0^\pi 2c \sin x dx = -2c\cos x|_0^\pi = -2c(-1-1) = 4c,$$

$$\int_0^\pi \sin^2 x dx = \int_0^\pi \frac{1-\cos(2x)}{2} dx = (\frac{x}{2} - \frac{1}{4}\sin(2x))|_0^\pi = \frac{\pi}{2}.$$

Therefore

$$V(c) = \pi(\frac{\pi}{2} - 4c + \pi c^2).$$

To find the minimal and maximal values of c, we need to compute the values of V at the critical points and at the endpoints. Since $V'(c) = \pi(-4 + 2\pi c)$, the only critical point is given by the equation $V'(c) = 0 \Leftrightarrow c = 2/\pi$. We have:

$$V(0) = \pi^2/2 \approx 4.93, \ V(1) = \pi(\frac{\pi}{2} - 4 + \pi) = \pi(3\pi/2 - 4) \approx 2.23,$$

$$V(2/\pi) = \pi(\frac{\pi}{2} - 8/\pi + 2\pi/\pi^2) = \pi(\pi/2 - 6/\pi) \approx 0.93.$$

Therefore the minimal volume is at $c = 2/\pi$ and the maximal volume is at $c = 0$.

63. (10 points) Consider the region R bounded by the graphs of $y = f(x) > 0$, $x = a$, $x = b$ and $y = 0$. If the volume of the solid formed by revolving R about the x-axis is 4π and the volume of the solid formed by revolving R about the line $y = -1$ is 8π, find the area of R.

Solution: We have

$$\int_a^b \pi f(x)^2 dx = 4\pi, \ \int_a^b \pi((f(x)+1)^2 - 1^2)dx = 8\pi,$$

so

$$\int_a^b f(x)^2 dx = 4, \ \int_a^b (f^2(x) + 2f(x) + 1 - 1)dx = \int_a^b (f^2(x) + 2f(x))dx = 8.$$

Therefore $\int_a^b 2f(x)dx = 4$, and the area of R equals $\int_a^b f(x)dx = 2$.

Section 6.2: 6.(10 points) Find the volume of the solid generated by revolving around the y-axis the region bounded by the graph of $y = \frac{9x}{\sqrt{x^3+9}}$, $y = 0$ and $x = 3$.

Solution: By the shell method we get

$$V = \int_0^3 2\pi x \cdot \frac{9x}{\sqrt{x^3+9}} dx = \int_0^3 \frac{18\pi x^2}{\sqrt{x^3+9}} dx.$$

Let $u = x^3 + 9$, then $du = 3x^2 dx$, so

$$V = \int_9^{36} \frac{6\pi du}{\sqrt{u}} = 12\pi\sqrt{u}|_9^{36} = 12\pi(6-3) = 36\pi.$$

Section 6.3: 26.(10 points) The astroid is given by the equation

$$x^{2/3} + y^{2/3} = 1.$$

Find the length of the astroid.

Solution: In the first quadrant the astroid is given by the equation

$$y^{2/3} = 1 - x^{2/3} \ \Leftrightarrow \ y = (1 - x^{2/3})^{3/2}.$$

By Chain Rule

$$y' = \frac{3}{2}(1 - x^{2/3})^{1/2} \cdot (-\frac{2}{3}x^{-1/3}) = -x^{-1/3}(1 - x^{2/3})^{1/2}.$$

Now

$$1 + (y')^2 = 1 + x^{-2/3}(1 - x^{2/3}) = 1 + x^{-2/3} - 1 = x^{-2/3}.$$

The length of one quarter of the astroid equals

$$\int_0^1 \sqrt{1 + (y')^2} dx = \int_0^1 x^{-1/3} dx = \frac{3}{2} x^{2/3}|_0^1 = \frac{3}{2}.$$

The total length of the astroid equals $4 \cdot \frac{3}{2} = 6$.