Lecture 1 (1/8)

Wednesday, January 3, 2024 6:44 PM

What is this class about?

$\mathbb{K}$ = field $\quad \mathbb{A}^n = \mathbb{A}^n(\mathbb{K}) = \{(x_1, \dots, x_n) : x_i \in \mathbb{K}\}$

n-dimensional space over $\mathbb{K}$.

$f_1(x_1 \dots x_n), \dots, f_m(x_1 \dots x_n)$ = some polynomials in $\mathbb{K}[x_1, \dots, x_n]$

Def An <u>affine algebraic variety</u> (= algebraic set) is

$$Z = \{f_1(x_1 \dots x_n) = \dots = f_m(x_1 \dots x_n) = 0\} \subset \mathbb{A}^n$$

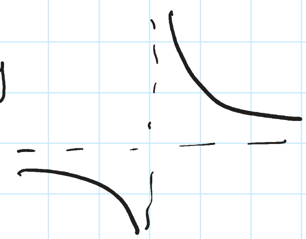
defined by system of polynomial equations.

Ex $Z = \{f(x, y) = 0\} \subset \mathbb{A}^2(\mathbb{K})$

(1) $\mathbb{K} = \mathbb{R} \Rightarrow$ one equation in $\mathbb{R}^2$, typically a 1d curve.

$\{x^2 + y^2 - 1 = 0\}$ circle

$\{xy - 1 = 0\}$ hyperbola

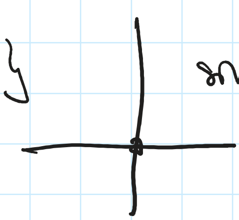


! $\{x^2 + y^2 + 1 = 0\}$ $\emptyset$

Could be empty!

Q: Is it compact? Connected? How many connected comp.? Smooth? ← 239

$\{xy = 0\}$ singular



(2) $\mathbb{K} = \mathbb{C} \Rightarrow$ one (complex eq.) in $\mathbb{C}^2$, typically a

(2) $\mathbb{K} = \mathbb{C} \Rightarrow$ one (complex eq.) in $\mathbb{C}^2$, typically a 2d surface

noncompact Smooth $\Rightarrow$ genus g surface w. boundary

<u>Q</u>: How to read the genus g from the equation?

(3) $\mathbb{K} = \mathbb{F}_q \Rightarrow \{f(x,y)=0\}$ is finite set

$\#\{x^2+y^2-1=0\}$ in $\mathbb{F}_q$?

(4) $\mathbb{K} = \mathbb{Q} \Rightarrow$ <u>rational</u> points in Z.

Number theory

<u>Fact</u> let g = genus of <u>complex</u> curve $\{f(x,y)=0\}$

- If $g=0$, either 0 or ∞ many $\mathbb{Q}$ points (<u>Easy</u>)
- If $g=1$, $\mathbb{Q}$ points = finitely generated abelian group (Mordell's Thm). Hard
- If $g>1$, finitely many $\mathbb{Q}$ points (Faltings' Thm, very hard).

Now generalize this to higher dimensions / more general eqns.

Can we say anything about geometry / topology of Z from the equations?

<u>Def</u> Projective space $\mathbb{P}^n(\mathbb{K}) = \{[z_0 : z_1 : \dots : z_n]\}/\sim$

Not all $z_i = 0$ $\quad [z_0 : \dots : z_n] \sim [\lambda z_0 : \dots : \lambda z_n]$

Not all $z_i = 0$, $[z_0 : \dots : z_n] \sim [\lambda z_0 : \dots : \lambda z_n]$, $\lambda \neq 0$

Fact/exercise $\mathbb{P}^n(\mathbb{K}) = \{$ all lines in $\mathbb{A}^{n+1}(\mathbb{K})$ through $0 \}$.

If $f(z_0, \dots, z_n)$ is a homogeneous polynomial of degree d, then $f(\lambda z_0, \dots, \lambda z_n) = \lambda^d f(z_0, \dots, z_n)$

Therefore $\{f(z_0, \dots, z_n) = 0\}$ is a well defined subset of $\mathbb{P}^n$.

Def A projective algebraic variety is the set $\{f_1(z_0, \dots, z_n) = \dots = f_m(z_0, \dots, z_n) = 0\} \subset \mathbb{P}^n$

Same questions as above.

Now more formally | $\mathbb{K} =$ field $\quad A = \mathbb{K}[x_1, \dots, x_n]$

$T \subset A$ collection of polynomials (not necessarily finite)

Def $Z(T) = \{(x_1, \dots, x_n) \mid f(x_1, \dots, x_n)$ for all $f \in T\} \subset \mathbb{A}^n(\mathbb{K})$

All such Z are called algebraic sets.

Lemma 1) $T_1 \subset T_2 \Rightarrow Z(T_1) \supset Z(T_2)$

2) $I =$ ideal generated by $T \Rightarrow Z(I) = Z(T)$.

Proof: 1) $p \in Z(T_2) \Rightarrow f(p) = 0$ for all f in T_2

$\Rightarrow f(p) = 0$ for all f in T_1.

$\Rightarrow f(p)=0$ for all f in T_1.

2) Any element of I has the form $\sum f_i g_i$, $f_i \in T$, g_i any.
Given $p \in Z(T)$, we have $f_i(p)=0$

$$\Rightarrow \sum f_i(p) g_i(p) = 0, \text{ so } Z(T) \subset Z(I).$$

On the other hand, $T \subset I \Rightarrow Z(I) \subset Z(T)$. ☒

I_1, I_2 = ideals

<u>Lemma</u> (a) $Z(I_1+I_2) = Z(I_1) \cap Z(I_2)$

(b) $Z(I_1 \cap I_2) = Z(I_1) \cup Z(I_2)$

(c) $Z(1) = \emptyset$ (d) $Z(0) = \mathbb{A}^n$

<u>Pf</u> (a), (c), (d) clear by defn. since I_1+I_2 = ideal gen. by $I_1 \cup I_2$.

(b): Assume $p \in Z(I_1) \cup Z(I_2)$.

Then either $p \in Z(I_1)$ or $p \in Z(I_2)$

Given $f \in I_1 \cap I_2$, f belongs both to p_1 and p_2

$\Rightarrow f(p)=0$. So $Z(I_1) \cup Z(I_2) \subset Z(I_1 \cap I_2)$.

Conversely, assume $p \in Z(I_1 \cap I_2)$ but $p \notin Z(I_1)$, $p \notin Z(I_2)$.

Then there exist $f \in I_1$, $g \in I_2$ such that

$f(p) \neq 0$, $g(p) \neq 0$. Then $f(p)g(p) \neq 0$.

On the other hand $fg \in I_1 \cap I_2$ since I_1, I_2 are ideals.

Contradiction. ☒
