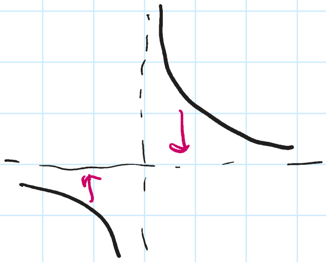
Lecture 10 (1/31)

Wednesday, January 31, 2024 7:49 AM

Recall: $\left\{ \begin{array}{c} \text{morphisms} \\ X \xrightarrow{\varphi} Y \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{algebra} \\ \text{homomorphisms} \\ A(Y) \xrightarrow{\varphi^*} A(X) \end{array} \right\}$

Ex



$X = \{xy = 1\} \subset \mathbb{A}^2$

$Y = \mathbb{A}^1_x$ $\qquad \varphi : X \to Y$

$\varphi(x, y) = x$

Note: $\varphi(X)$ is not closed in Y, so φ is not a closed map.

$$A(X) = \frac{\mathbb{K}[x,y]}{xy-1} = \mathbb{K}[x, x^{-1}] \xleftarrow{\varphi^*} A(Y) = \mathbb{K}[x]$$

φ^* = inclusion of $\mathbb{K}[x]$ into $\mathbb{K}[x, x^{-1}]$, injective.

Def A morphism $\varphi : X \to Y$ is called dominant, if the image of X is dense in Y, that is, $\overline{\varphi(X)} = Y$.

Thm $\varphi : X \to Y$ is dominant iff $\varphi^* : A(Y) \to A(X)$ is injective.

Pf ① Assume φ^* is injective, need to prove $\overline{\varphi(X)} = Y$.

Suppose $g \in A(Y)$ vanishes on $\varphi(X)$.

Then $\varphi^* g = g \circ \varphi = 0$ in $A(X) \Rightarrow$ since φ^* is injective we get $g = 0$.

Therefore the ideal of $\overline{\varphi(X)}$ in $A(Y) = 0$

$\Rightarrow \overline{\varphi(X)} = Y$.

② Assume φ^* is not injective and $\varphi^* g = 0$ for some $g \neq 0$.

(2) Assume φ^* is not injective and $\varphi^* g = 0$ for some $g \neq 0$.

Then $g(\varphi(x)) = 0$ for all $x \in X$ and

$$\varphi(X) \subset \{g = 0\} \subsetneq Y. \quad \text{Note: } \{g=0\} \text{ closed.}$$

$\Rightarrow \overline{\varphi(X)} \subset \{g=0\} \neq Y$, so φ is not dominant. ■

Thm φ^* surjective $\iff$ $\varphi(X)$ closed and φ isomorphism onto its image. (closed embedding)

Pf: (1) Assume φ^* surjective, $Z = \overline{\varphi(X)}$ closed in Y.

We get a diagram of maps

$$\begin{tikzcd} X \arrow[d, "\alpha"'] \arrow[dr, "\varphi"] & \\ Z \arrow[r, hook] & Y \end{tikzcd} \qquad \begin{tikzcd} A(X) & \\ A(Z) \arrow[u, hook, "\alpha^*"] & A(Y) \arrow[l, two heads] \arrow[ul, "\varphi^*"'] \end{tikzcd}$$

injective since $\alpha: X \to Z$ dominant

surjective

$A(Z) = \dfrac{A(Y)}{J}$

$A(Y) \twoheadrightarrow A(Z)$ surjective

Therefore $\alpha^*: A(Z) \to A(X)$ surjective and injective $\Rightarrow$ isomorphism.

$\Rightarrow \alpha$ = isomorphism. and $Z = \overline{\varphi(X)} = \varphi(X)$.

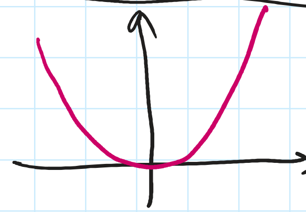
(2) Assume φ iso onto its image, $Z = \varphi(X)$. Then again we get

$$\begin{tikzcd} X \arrow[d, "\wr"'] \arrow[dr, "\varphi"] & \\ Z \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} A(X) & \\ A(Z) \arrow[u, "\wr"] & A(Y) \arrow[l, two heads] \arrow[ul, "\varphi^*"'] \end{tikzcd}$$

So φ^* is surjective. ■

Ex. $X = \mathbb{A}^1$, $Y = \mathbb{A}^2_{x,y}$, $\varphi(t) = (t, t^2)$

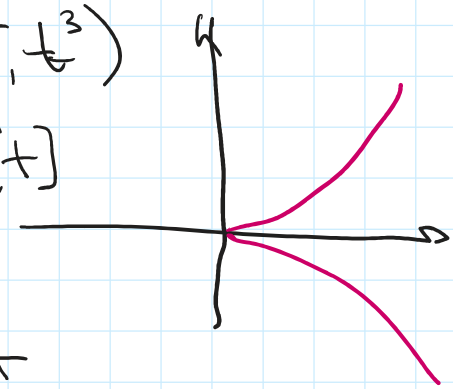
Ex. $X = \mathbb{A}^1_t$, $Y = \mathbb{A}^2_{x,y}$, $\varphi(t) = (t, t^2)$



$\varphi^*(x) = t$, $\varphi^*(y) = t^2 \Rightarrow \varphi^*$ surjective

$\varphi(X) =$ parabola in $\mathbb{A}^2$.

Ex $X = \mathbb{A}^1_t$, $Y = \mathbb{A}^2_{x,y}$, $\varphi(t) = (t^2, t^3)$



$\varphi^* x = t^2$ $\quad \varphi^* A(Y) = \mathbb{K}[t^2, t^3] \neq \mathbb{K}[t]$

$\varphi^* y = t^3$ $\quad \varphi^*$ not surjective.

Rmk φ injective, $\varphi(X)$ closed but φ is NOT an isomorphism in this example

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What if X, Y are not affine alg. sets?

Need to work locally.

Warm-up $X =$ alg. set, $g \in A(X)$

$D(g) = \{g \neq 0\}$ open in X ("principal open subset")

Def A regular function on $D(g)$ is a rational function of the form $\dfrac{f}{g^k}$, $k \geq 0$.

Note: $\dfrac{f}{g^k} = \dfrac{fg}{g^{k+1}}$

These form a ring: $\dfrac{f_1}{g^{k_1}} + \dfrac{f_2}{g^{k_2}} = \dfrac{f_1 g^{k_2} + f_2 g^{k_1}}{g^{k_1+k_2}}$

$\dfrac{f_1}{g^{k_1}} \cdot \dfrac{f_2}{g^{k_2}} = \dfrac{f_1 f_2}{g^{k_1+k_2}}$

$$\frac{f_1}{g^{k_1}} \cdot \frac{f_2}{g^{k_2}} = \frac{f_1 f_2}{g^{k_1+k_2}}$$

Thm Consider $Z \subset X \times \mathbb{A}^1_t$ defined by the equation $\{g(\bar{x}) \cdot t = 1\}$. Then $Z \simeq D(g)$.

In other words, $D(g)$ = open in X is isomorphic to closed subset of $X \times \mathbb{A}^1$.

Pf ① Geometrically:

$$D(g) \xrightarrow{\varphi_1} Z \qquad\qquad Z \xrightarrow{\varphi_2} D(g)$$

$$x \longrightarrow \left(x, \underbrace{\frac{1}{g(x)}}_{\text{ok since this is regular on } D(g)}\right) \qquad\qquad (x,t) \longrightarrow x.$$

Clearly, these are inverse to each other, since $t = \frac{1}{g}$.

② Algebraically:

$$A(Z) = \frac{A(X)[t]}{(g(x) \cdot t - 1)} \longrightarrow A(D(f)) = \left\{\frac{f}{g^k}\right\}$$

$$f \cdot t^k \longmapsto \frac{f}{g^k}$$

$$g(x) \cdot t - 1 \longmapsto g(x) \cdot \frac{1}{g(x)} - 1 = 0 \quad \text{ok.}$$

Conversely, $\frac{f}{g^k} \longrightarrow f t^k$.

$$\frac{f_1}{g^{k_1}} = \frac{f_2}{g^{k_2}} \text{ iff } \overset{k_1 < k_2}{} f_1 g^{k_2} - f_2 g^{k_1} = 0 \Leftrightarrow \boxed{f_1 g^{k_2 - k_1} - f_2 = 0}$$

$$\frac{f_1}{g^{k_1}} = \frac{f_2}{g^{k_2}} \quad \text{iff} \quad f_1 g^{k_2} - f_2 g^{k_1} = 0 \iff \boxed{f_1 g^{k_2-k_1} - f_2 = 0}$$

$$\frac{f_1}{g^{k_1}} \mapsto f_1 \cdot t^{k_1} \qquad \frac{f_2}{g^{k_2}} \mapsto f_2 t^{k_2}$$

$$f_1 \cdot t^{k_1} - f_2 t^{k_2} = t^{k_1}(f_1 - f_2 t^{k_2-k_1})$$
$$= t^{k_1}(f_1 - f_1 \cdot g^{k_2-k_1} \cdot t^{k_2-k_1})$$
$$= t^{k_1} f_1 (1 - g^{k_2-k_1} \cdot t^{k_2-k_1})$$

and $1 - (gt)^{k_2-k_1}$ is divisible by $1 - gt$. ∎