Thm Consider $Z \subset X \times \mathbb{A}^1_t$ defined by the equation $\{g(\bar{x}) \cdot t = 1\}$. Then $Z \simeq D(g)$.

In other words, $D(g)$ = open in X is isomorphic to closed subset of $X \times \mathbb{A}^1$.

Pf (1) Geometrically:

$$\begin{array}{ccc} D(g) & \xrightarrow{\varphi_1} & Z \\ x & \longrightarrow & \left(x, \frac{1}{g(x)}\right) \end{array} \qquad \left\{ \begin{array}{ccc} Z & \xrightarrow{\varphi_2} & D(g) \\ (x,t) & \longrightarrow & x. \end{array} \right.$$

ok since this is regular on $D(g)$

Clearly, these are inverse to each other, since $t = \frac{1}{g}$.

(2) Algebraically:

$$A(Z) = \frac{A(X)[t]}{(g(x) \cdot t - 1)} \longrightarrow A(D(f)) = \left\{ \frac{f}{g^k} \right\}$$

$$f \cdot t^k \longmapsto \frac{f}{g^k}$$

$$g(x) \cdot t - 1 \longmapsto g(x) \cdot \frac{1}{g(x)} - 1 = 0$$

ok.

Conversely, $\frac{f}{g^k} \longrightarrow f t^k$.

$$\frac{f_1}{g^{k_1}} = \frac{f_2}{g^{k_2}} \text{ iff } f_1 g^{k_2} - f_2 g^{k_1} = 0 \Leftrightarrow \boxed{f_1 g^{k_2 - k_1} - f_2 -}$$

$k_1 < k_2$

$g^{k_1} \quad g^{k_2}$ iff $\quad f_1 g \quad - f_2 g = 0 \iff f_1 g \quad - f_2$

$f_1 \cdot t^{k_1} \qquad f_2 t^{k_2}$

$$f_1 \cdot t^{k_1} - f_2 t^{k_2} = t^{k_1}(f_1 - f_2 t^{k_2 - k_1})$$

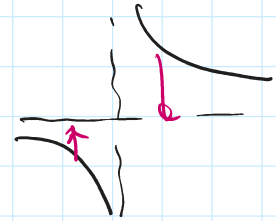
$$= t^{k_1}(f_1 - f_1 \cdot g^{k_2-k_1} \cdot t^{k_2-k_1})$$

and

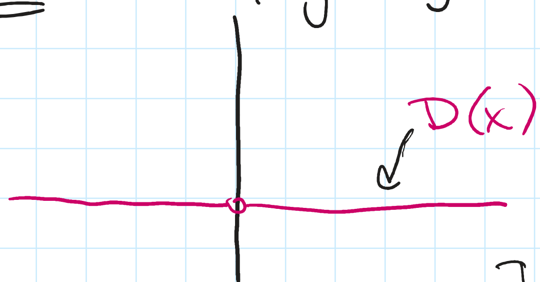
$$= t^{k_1} f_1 (1 - g^{k_2-k_1} \cdot t^{k_2-k_1})$$

$1 - (gt)^{k_2-k_1}$ is divisible by $1 - gt$. $\blacksquare$

Ex $\underset{\text{open}}{\{x \neq 0\} \subset \mathbb{A}^1} \xleftrightarrow{\sim} \underset{\text{closed}}{\{xt = 1\} \subset \mathbb{A}^2}$



Ex $X = \{xy = 0\} \supset D(x) = \{x \neq 0\}$



In the localization on $D(x)$:

$$y = \frac{xy}{x} = \frac{0}{x} = 0$$

Indeed, if $xy = 0$ and $x \neq 0$ then $y = 0$.

$Z \subset X \times \mathbb{A}^1$, $Z = \{xy = 0, xt = 1\} \subset \mathbb{A}^3$

$$y = yxt = 0$$

Note: x = zero divisor!

Lemma Suppose X is an irreducible affine set. Then $A(X)$ is a domain, that is, there are no zero divisors.

Pf X irreducible $\iff I(X)$ prime

$A(X) = \frac{\mathbb{k}[x_1, \ldots, x_n]}{I(X)} \qquad fg = 0$ in $A(X) \iff fg \in I(X)$

$$A(X) = \frac{\mathbb{K}[x_1 \ldots x_n]}{I(X)} \qquad fg = 0 \text{ in } A(X) \iff fg \in I(X)$$

Since $I(X)$ is prime, either f or $g \in I(X)$. ☒

Cor If X is irreducible, then for $g \neq 0$

$$A(X) \hookrightarrow A(D(g)) = A(X)[g^{-1}]$$

is injective. **Note** $D(g)$ = open in $X \Rightarrow$ dense since X is irreducible.

Def W is called quasi-affine if it isomorphic to an open subset in an affine algebraic set.

Q: How to describe functions on W? $\boxed{\mathcal{U}(W) = \text{ring of regular functions}}$

A1: $W \underset{\text{open}}{\subseteq} X$, X = alg. set

Recall: $W = \bigcup_i D(g_i)$ for some function g_i

A function $\varphi : W \longrightarrow \mathbb{K}$ is regular, if $\varphi|_{D(g_i)}$ is regular for all i.

Ex: $W = \mathbb{A}^2 \setminus \{(0,0)\} = D(x) \cup D(y)$

either $x \neq 0$ or $y \neq 0$

$\varphi : W \to \mathbb{K}$ regular $\quad \varphi|_{D(x)} = \dfrac{f_1}{x^k} \quad \varphi|_{D(y)} = \dfrac{f_2}{y^m}$

So $\dfrac{f_1}{x^k} = \dfrac{f_2}{y^m} \Rightarrow f_1 y^m = f_2 x^k$

Since $GCD(x, y) = 1$ and $\mathbb{K}[x, y]$ is UFD, we conclude

Since $GCD(x,y)=1$ and $\mathbb{K}[x,y]$ is UFD, we conclude that f_1 is divisible by x^k, f_2 is divisible by y^m

$\Rightarrow \varphi = \frac{f_1}{x^k}$ is <u>a polynomial</u>.

So: Any regular function on $W = \mathbb{A}^2 \setminus \{(0,0)\}$ extends to a regular function on $\mathbb{A}^2$. ("algebraic Hartogs lemma")

Ex $\underbrace{\{xy = z^2\}}_{= X} \subset \mathbb{A}^3$  $\frac{x}{z} = \frac{z}{y}$ in localization of $A(X)$

$\Rightarrow \varphi = \frac{x}{z}$ is regular both on $\{z \neq 0\}$ and $\{y \neq 0\}$.

$W = D(y) \cup D(z) = X \setminus \{y = z = 0\} = \{\text{cone} \setminus \text{line}\}$

φ is regular on W.

A2 X = alg. set, $W \underset{\text{open}}{\subset} X \underset{\text{closed}}{\subset} \mathbb{A}^n$. $\varphi : W \to K$

Def

- φ is <u>regular at a point</u> $P \in W$ if there exists an open subset $p \in U \subset W$ and rational function $\frac{g}{h}$, $g, h \in \mathbb{K}[x_1 \ldots x_n]$, such that ① $h \neq 0$ on U
  ② $\varphi = \frac{g}{h}$ on U.

- φ is <u>regular on W</u> if it is regular at every point of W. $\boxed{\mathcal{O}(W) = \text{ring of regular functions.}}$

every point of W. $\boxed{\mathcal{O}(W) = \text{ring of regular functions.}}$

<u>Thm</u> a) Suppose W = alg. set, then $\mathcal{O}(W) = A(W)$

b) Suppose $W = D(g) \subset X$, then $\mathcal{O}(W) = A(X)[g^{-1}]$

c) Two definitions of $\mathcal{O}(W)$ agree.

<u>Proof</u>: a) Clearly, $A(W) \subset \mathcal{O}(W)$; Suppose $\varphi \in \mathcal{O}(W)$

Define $J = \{ h \in \mathbb{K}[x_1 \ldots x_n] : h\varphi \in A(W) \} + I(W)$

For all $p \in W$, $\exists h : h(p) \neq 0$, $h\varphi \in A(W) \Rightarrow h \in J$

Therefore $Z(J) = \emptyset \Rightarrow$ by Nullstellensatz

$J = \mathbb{K}[x_1 \ldots x_n] \Rightarrow 1 \in J$ and $\varphi \in A(W)$.

b) Follows from (a) and $D(g) \simeq \{ g t = 1 \}$

c) Assume $W = \bigcup_i D(g_i) \subset X$

- If $\varphi|_{D(g_i)}$ is regular for all i, then $\varphi|_{D(g_i)} = \dfrac{f}{g_i^k}$ and φ is regular at all points of $D(g_i)$ by 2nd definition.

  $\Rightarrow \varphi$ is regular at all points of W.

- Suppose φ is regular at all points of W.

  Given $p \in D(g_i)$, $\exists U \subset W$ open such that $\varphi|_U = \dfrac{f}{h}$, $h \neq 0$ on U. We can restrict φ to $U \cap D(g_i)$ = open

  $\Rightarrow$ by part (b) $\varphi|_{D(g_i)} = \dfrac{f}{g_i^k}$ and we are done.

$\Rightarrow$ by part (b) $\varphi|_{D(g_i)} = \frac{t}{g_i^k}$ and we are done.