Lecture 12 (2/5)

Monday, February 5, 2024 8:55 AM

Recall

Def $W \subset \mathbb{A}^n$ quasi-affine variety

1) $\varphi : W \to \mathbb{K}$ is regular at p if $\exists$ open $U \subset W$, $p \in U$ and $f, g \in \mathbb{K}[x_1 \dots x_n]$ such that $g \neq 0$ on U and $\varphi = \frac{f}{g}$ on U.

2) φ is regular if it is regular at all points.

3) $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$, $\Phi : V \to W$ if $\Phi = (\varphi_1 \dots \varphi_m)$, φ_i = regular, that is, for all p there is an open $U \ni p$ such that $\Phi = \left(\frac{f_1}{g_1} \dots \frac{f_m}{g_m}\right)$ and $g_i \neq 0$ on U.

Def $W \subset \mathbb{P}^n$ quasi-projective (open in projective)

1) $\varphi : W \longrightarrow \mathbb{K}$ regular at p if there is an open $U \ni p$ such that $\varphi = \frac{f}{g}$ on U, $g \neq 0$ on U and f, g = homogeneous polynomials of the same degree in homogeneous coordinates.

Note: $\varphi(\lambda x_0 : \dots : \lambda x_n) = \dfrac{f(\lambda x_0, \dots \lambda x_n)}{g(\lambda x_0, \dots \lambda x_n)} = \dfrac{\lambda^d f(x_0 \dots x_n)}{\lambda^d g(x_0 \dots x_n)} = \dfrac{f}{g} = \varphi$

So φ is a well defined function on $\{g \neq 0\} \subset \mathbb{P}^n$.

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3) $V \subset \mathbb{P}^n$, $W \subset \mathbb{P}^m$, $\Phi: V \to W$ regular at p if $\exists$ open $\mathcal{U} \ni p$ such that

$$\Phi(x_0 : \ldots : x_n) = [\varphi_0(\bar{x}), \ldots \varphi_m(\bar{x})]$$

where φ_i are homogeneous polynomials in x_i of the same degree and φ_i do not vanish simultaneously on $\mathcal{U}$.

Notes: ① If $\varphi_i(\bar{x}) \neq 0$, we can rewrite

$$[\varphi_0(\bar{x}) : \ldots : \varphi_m(\bar{x})] = \left[\frac{\varphi_0}{\varphi_i} : \ldots \frac{\varphi_i}{\varphi_i} : \frac{\varphi_m}{\varphi_i}\right]$$ ← homog. same degree

so we get a well-defined regular map

$$\Phi : \mathcal{U} \longrightarrow \mathbb{A}^m = \{\varphi_i \neq 0\}.$$

② If Φ is given by rational functions

$$\Phi = \left[\frac{f_0}{g_0} : \ldots : \frac{f_m}{g_m}\right]$$ then we can clear denominators.

$\deg f_i = \deg g_i = d_i$

$$\Phi = [f_0 g_1 \ldots g_m : \ldots : f_m g_0 \ldots g_{m-1}]$$

all these are polynomials of same degree $\sum d_i$.

Ex $C = \{xy = z^2\} \subset \mathbb{P}^2$ w. coords $[x : y : z]$.

① Define map $\mathbb{P}^1 \xrightarrow{\Phi} C$

by $[a : b] \longrightarrow [a^2 : b^2 : ab]$ ← homog. degree 2

Since $a^2 \cdot b^2 = (ab)^2$, $\Phi(\mathbb{P}^1) \subset C$.

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(2) Define map $C \xrightarrow{\Psi} \mathbb{P}^1$

$$[x:y:z] \xrightarrow{\Psi} \begin{cases} [x:z] & \text{if } x \neq 0 \\ [z:y] & \text{if } y \neq 0. \end{cases}$$

Note that if $x=y=0$, then $z=0$ and $[x:y:z]$ is not in $\mathbb{P}^2$

Also, whenever $x \neq 0$ and $y \neq 0$ we get $z \neq 0$, so

$$[x:z] = \left[\frac{x}{z} : 1\right] \underset{\uparrow}{=} \left[\frac{z}{y} : 1\right]$$

since $xy = z^2$

So the morphism $\Psi : C \to \mathbb{P}^1$ is well defined

on $\{x \neq 0\} \cap \{y \neq 0\}$.

Exercise Φ and Ψ are inverse to each other, so $C \simeq \mathbb{P}^1$.

Lemma Any regular function on $\mathbb{P}^n$ is a constant. (everywhere).

Proof Assume $\varphi : \mathbb{P}^n \to \mathbb{K}$ is regular

Define $\tilde{\varphi} : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{K}$ by the same formula.

Then $\tilde{\varphi}$ is regular on $\mathbb{A}^{n+1} \setminus \{0\}$ (prove it!)

$\Rightarrow$ by the result of last lecture $\tilde{\varphi} \in \mathbb{K}[x_0, \dots, x_n]$

is a polynomial. But $\tilde{\varphi}(x_0, \dots x_n) = \tilde{\varphi}(\lambda x_0, \dots \lambda x_n)$

is a polynomial. But $\tilde{\varphi}(x_0, \dots x_n) = \tilde{\varphi}(\lambda x_0, \dots \lambda x_n)$

$\Rightarrow \tilde{\varphi}$ is a constant.

In fact, much stronger statement is true.

Thm $X \subset \mathbb{P}^n$ irreducible projective alg. set.
Then any global function on X is a constant.

Proof: Assume $\varphi : X \to \mathbb{K}$ is regular.

Cover $\mathbb{P}^n$ by affine charts $U_i = \{x_i \neq 0\}$

$X \cap U_i$ = affine alg. subset $\subset U_i = \mathbb{A}^n$

$\Rightarrow \varphi|_{X \cap U_i} = g\left(\frac{x_0}{x_i}, \dots \frac{x_n}{x_i}\right)$ polynomial of degree N_i

$\Rightarrow x_i^{N_i} \varphi$ is a polynomial $(\text{mod } I(X))$ for all i.

Pick $N \gg \sum N_i$, then $G \cdot \varphi$ is a polynomial of degree N
for any monomial G of degree N.

$\Rightarrow G \cdot \varphi^k$ is a polynomial of degree N
for any monomial G of degree N.

Then $\frac{\mathbb{K}[x]}{I(X)} \subset \frac{\mathbb{K}[x]}{I(X)} \oplus \varphi \frac{\mathbb{K}[x]}{I(X)} \subset \frac{\mathbb{K}[x]}{I(x)} + \varphi \frac{\mathbb{K}[x]}{I(x)} + \varphi^2 \mathbb{K}[x] \subset \dots \subset G^{-1} \frac{\mathbb{K}[x]}{I(x)}$.

Since $\frac{\mathbb{K}[x]}{I(X)}$ is Noetherian, this eventually stabilizes

and $\varphi^k = a_{k-1}\varphi^{k-1} + \dots + a_0$ for some

(*) [illegible]

and $\varphi^k = a_{k-1}\varphi^{k-1} + \dots + a_0$ for some
(*) mod $I(X)$ $a_i \in \mathbb{K}[x_1 \dots x_n]$

But φ has degree 0 $\Rightarrow$ we can just consider the homogeneous component of (*) of degree 0:

$$\varphi^k = a_{k-1}(0)\varphi^{k-1} + \dots + a_0(0)$$

$\Rightarrow$ φ is a solution of a monic polynomial w. constant coefficients. Since $\mathbb{K}$ is alg. closed,

$$\prod_i (\varphi - r_i) = 0 \mod I(X).$$
$r_i \in \mathbb{K}$

$\Rightarrow X = \bigcup_i \{\varphi = r_i\}$. Since X is irreducible, $\varphi \equiv r_i$ for some $i \Rightarrow \varphi$ is constant. □

<u>Rmk</u> The condition that X is irreducible is important: for example, any finite set of points is a projective alg. set (prove it!) and we can choose different functions at different points.

<u>Rmk</u> In complex analysis, Liouville's Thm says that whenever $\varphi: X \to \mathbb{C}$ is holomorphic and X is a <u>compact</u>, connected complex manifold then φ is constant. Any closed subset of $\mathbb{P}^n$ is compact.

φ is constant. Any ~~closed~~ connected subset of $\mathbb{R}^n$ is compact.