Lecture 13 (2/6)

Tuesday, February 6, 2024 6:50 PM

Blow-up

$X \subset \mathbb{A}^2 \times \mathbb{P}^1$ consists of pairs (p, ℓ) such that
$\phantom{X \subset \mathbb{A}^2} p \quad\ \ \ell$

(1) $p \in \mathbb{A}^2$ point

(2) $\ell \in \mathbb{P}^1 \longleftrightarrow$ line in $\mathbb{A}^2$ through 0

(3) $p \in \ell$.

Claim: X is a closed subset of $\mathbb{A}^2 \times \mathbb{P}^1$

Pf: $p = (x, y) \quad \ell = [u : v]$

$p \in \ell$ if the vectors (x, y) and (u, v) are collinear,

so $\begin{vmatrix} x & y \\ u & v \end{vmatrix} = 0$ and $\boxed{xv = uy}$

Note: $xv - uy$ is a polynomial in x, y, and a homogeneous polynomial in $[u : v]$, so well defined.

We have 2 projections

$$\begin{tikzcd} & X \subset \mathbb{A}^2 \times \mathbb{P}^1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{A}^2 & & \mathbb{P}^1 \end{tikzcd}$$

What can we say about π_1, π_2 and their fibers?

(1) $\beta : X \longrightarrow \mathbb{P}^1 \qquad (p, \ell) \longrightarrow \ell$

In coordinates $(x, y, [u : v]) \longrightarrow [u : v]$

$\beta^{-1}(\ell) = \{ p : p \in \ell \} \simeq \mathbb{A}^1$ for all ℓ

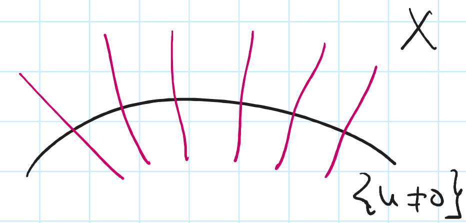
In coordinates (x,y), $[u:v]$, ...

$$\beta^{-1}(\ell) = \{ p : p \in \ell \} \simeq \mathbb{A}^1 \quad \text{for all } \ell.$$

We can say even more, if we work in charts on $\mathbb{P}^1$

- $\{u \neq 0\}$, $xv = uy \Rightarrow y = \frac{xv}{u}$

$$(x,y) = x \cdot \left(1, \frac{v}{u}\right)$$



__Claim:__ $\beta^{-1}(\{u \neq 0\}) \simeq \{u \neq 0\} \times \mathbb{A}^1$

__Proof:__ $(x,y,[u:v]) \longrightarrow [u:v], x$

"eliminate y"

$$\left(x, \frac{xv}{u}, [u:v]\right) \longleftarrow [u:v], x$$

Both maps are regular, inverse to each other.

- $\{v \neq 0\}$, $xv = uy \Rightarrow x = \frac{yu}{v}$

Similarly, $\beta^{-1}(\{v \neq 0\}) \simeq \{v \neq 0\} \times \mathbb{A}^1$.

__Conclusion:__ β = locally trivial fibration in Zariski topology.

(2) $\alpha : X \longrightarrow \mathbb{A}^2 \qquad (p, \ell) \longrightarrow p$

$$(x, y, [u:v]) \longrightarrow (x,y).$$

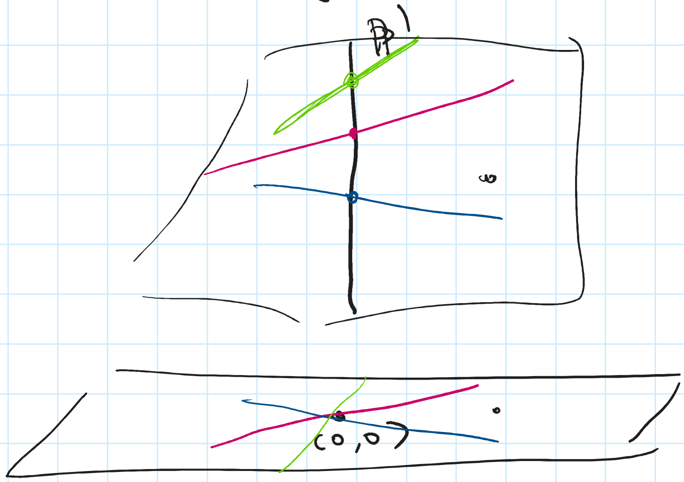
- If $(x,y) \neq (0,0)$, there exists a unique line ℓ through $(0,0)$ and $(x,y) \Rightarrow \alpha^{-1}(x,y) = $ one pt.

Algebraically: $xv = yu \Rightarrow [u:v] = [x:y]$ __if__ $(x,y) \neq (0,0)$

- If $(x,y) = (0,0)$, any line passes through $(0,0)$

$$\Rightarrow \alpha^{-1}(0,0) = \mathbb{P}^1 \longleftarrow \text{exceptional divisor}$$

$\Rightarrow \alpha^{-1}(0,0) = \mathbb{P}^1 \longleftarrow$ exceptional divisor E



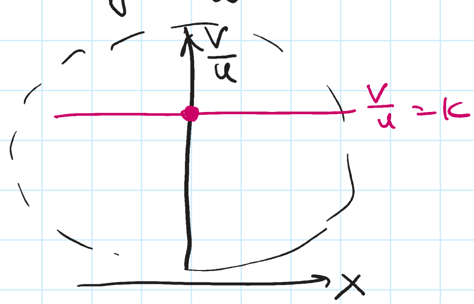
__Def__: X is called the blowup of $\mathbb{A}^2$ at $(0,0)$

What about α^{-1}(line in $\mathbb{A}^2$) $= \alpha^{-1}(y = kx)$

$$\begin{cases} xv = yu \\ y = kx \end{cases} \qquad [u;v] \in \mathbb{P}^1.$$

Let's work in charts:

- $\{u \neq 0\}$ $\quad y = \frac{xv}{u} = kx \Rightarrow x\frac{v}{u} = kx \Rightarrow x\left(\frac{v}{u} - k\right) = 0$



- $\{v \neq 0\}$ $\quad x = \frac{yu}{v}$, $y = kx \Rightarrow k\frac{yu}{v} = y$

$$y\left(\frac{u}{v} - \frac{1}{k}\right) = 0 \text{ if } k \neq 0$$

__Conclusion__: $\alpha^{-1}(\{y = kx\}) = \underset{\alpha^{-1}(0,0)}{\underset{\parallel}{E}} \cup \{\text{line } [u:v] = [1:k]\} \subset X$

__HW__: What about α^{-1}(other curves on $\mathbb{A}^2$)?

Q: Can we define maps back?

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$$\begin{tikzcd} & X \subset \mathbb{A}^2 \times \mathbb{P}^1 \arrow[dl, "\alpha"] \arrow[dr, "\beta"] & \\ \mathbb{A}^2 \arrow[ur, dashed, bend left, "\alpha'"] & & \mathbb{P}^1 \arrow[ul, dashed, bend right, "\beta'"'] \end{tikzcd}$$

Maybe want $\beta \circ \beta' = Id_{\mathbb{P}^1}$

$\alpha \circ \alpha' = Id_{\mathbb{A}^2}$

β': yes! "zero section" $[u:v] \longrightarrow (0,0,[u:v])$

$\beta'(\mathbb{P}^1) = E$ = exceptional divisor

α': no! $\alpha'(x,y) = \alpha^{-1}(x,y)$ for $(x,y) \neq (0,0)$

Cannot extend unique to $(0,0)$.

$$\alpha' : [\mathbb{A}^2 \setminus (0,0)] \longrightarrow X$$

$$(x,y) \longrightarrow \begin{cases} (x,y,[1:\frac{y}{x}]) & \text{if } x \neq 0 \\ (x,y,[\frac{x}{y}:1]) & \text{if } y \neq 0 \\ \text{undefined, if } & x=y=0. \end{cases}$$

Def: A rational map $\varphi : V \dashrightarrow W$ is a regular map $U \xrightarrow{\varphi} W$, where U open in V.

U = domain of φ

Equivalently, φ is locally given by rational functions and $U = \{\text{all denominators} \neq 0\}$.

Ex $\alpha : X \longrightarrow \mathbb{A}^2$ regular

$\alpha' : \mathbb{A}^2 \dashrightarrow X$ rational, defined on $\mathbb{A}^2 \setminus \{(0,0)\}$

Pedantic remark: Assume V irreducible. Then a rational map $\varphi : V \dashrightarrow W$ is an equivalence class of map $\varphi : V \dashrightarrow W$ is an equivalence class of pairs (U, φ) : U open in V, $\varphi : U \to W$ regular

$(U,\varphi) \sim (U', \varphi')$ if $\varphi = \varphi'$ on $U \cap U'$

Def V and W are called **birational** if

$\exists \quad V \underset{\psi}{\overset{\varphi}{\rightleftarrows}} W$ such that $\varphi \circ \psi = \psi \circ \varphi = Id$ wherever defined.

Ex X is birational to $\mathbb{A}^2$

More Generally:

$$Bl_0(\mathbb{A}^n) = \{ p \in \mathbb{A}^n, \ell \in \mathbb{P}^{n-1} : p \in \ell \}$$

blowup of $\mathbb{A}^n$ at $(0,0)$

Coords $(x_1, \dots x_n, [u_1 : \dots : u_n])$

Equations: $rk\begin{pmatrix} x_1 & \dots & x_n \\ u_1 & \dots & u_n \end{pmatrix} = 1 \Rightarrow \underset{\text{all}}{} \begin{vmatrix} x_i & x_j \\ u_i & u_j \end{vmatrix} = 0$

$\binom{n}{2}$ equations in $\mathbb{A}^n \times \mathbb{P}^{n-1}$.

Two projections:

$$\begin{array}{ccc} & Bl_0 \mathbb{A}^n & \\ \swarrow^{\alpha} & & \searrow^{\beta} \\ \mathbb{A}^n & & \mathbb{P}^{n-1} \end{array}$$

① $\beta^{-1}(\ell) \simeq \mathbb{A}^1$, more generally $\beta^{-1}(\{u_i \neq 0\}) \simeq \{u_i \neq 0\} \times \mathbb{A}^1$

(prove it!) $\Rightarrow \beta$ = locally (Zariski) trivial fibration over $\mathbb{P}^{n-1}$ with fiber $\mathbb{A}^1$

② $\alpha^{-1}(p) = \begin{cases} \text{one point}, & p \neq (0, \ldots, 0) \\ \underset{\displaystyle E}{\underset{\shortparallel}{\mathbb{P}^{n-1}}}, & p = (0, \ldots, 0) \end{cases}$

$(Bl_0 \mathbb{A}^n \setminus E) \simeq \mathbb{A}^n \setminus (0 \ldots 0)$

So $Bl_0 \mathbb{A}^n$ and $\mathbb{A}^n$ are <u>birational</u>.