Def: A rational map $\varphi: V \dashrightarrow W$ is a regular map $U \xrightarrow{\varphi} W$, where U open in V. U = domain of φ

Equivalently, φ is locally given by rational functions and $U = \{\text{all denominators} \neq 0\}$.

Ex $\alpha: X \longrightarrow \mathbb{A}^2$ regular ($X = Bl_0 \mathbb{A}^2$ from last lecture)

$\alpha': \mathbb{A}^2 \dashrightarrow X$ rational, defined on $\mathbb{A}^2 \setminus \{(0,0)\}$

Pedantic remark: Assume V irreducible. Then a rational map $\varphi: V \dashrightarrow W$ is an equivalence class of pairs (U, φ): U open in V, $\varphi: U \to W$ regular

$(U, \varphi) \sim (U', \varphi')$ if $\varphi = \varphi'$ on $U \cap U'$

Def V and W are called birational if

$\exists$ $V \underset{\psi}{\overset{\varphi}{\rightleftarrows}} W$ such that $\varphi \circ \psi = \psi \circ \varphi = Id$ wherever defined.

Ex X is birational to $\mathbb{A}^2$

Def A line bundle on X is the following data:

(1) A map $E \xrightarrow{\pi} X$

(2) The structure of 1-dim vector space in each

② The structure of 1-dim vector space in each fiber. That is, if $e_1, e_2 \in E$ and $\pi(e_1) = \pi(e_2) = x$ then we can define $\alpha e_1 + \beta e_2 \in \pi^{-1}(x)$ for $\alpha, \beta \in \mathbb{K}$.

③ local triviality: for each $x \in X$ there is an open $\mathcal{U} \ni x$ such that

$$\begin{tikzcd} \pi^{-1}(\mathcal{U}) \arrow[rr, "\sim"] \arrow[dr] & & \mathcal{U} \times \mathbb{A}^1 \arrow[dl] \\ & \mathcal{U} & \end{tikzcd}$$

Ex $X = \mathbb{P}^n$, $E = \{(p, \ell) : p \in \ell\} \simeq \mathrm{Bl}_0 \mathbb{A}^{n+1}$

$\pi : E \longrightarrow \mathbb{P}^n$. $\pi(p, \ell) = \ell$

This is called the tautological line bundle on $\mathbb{P}^n$ and denoted by $\mathcal{O}(-1)$.

Def More generally, we define:

for $k \geq 0$: $\mathcal{O}(k) = \{(f, \ell) : f$ is a degree k homogeneous polynomial on $\ell\}$

$\ell \in \mathbb{P}^n \simeq$ line on $\mathbb{A}^{n+1}$

for $k \leq 0$ $\mathcal{O}(-k) = \{(f, \ell) : f$ is a degree k homogeneous polynomial on $\ell^*\}$

for $k = 0$ we get $f \in \mathbb{K} \Rightarrow \mathcal{O}(0) \simeq \mathbb{A}^1 \times \mathbb{P}^n$. trivial bundle.

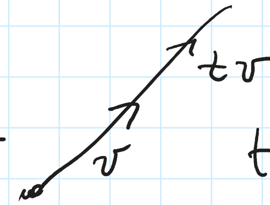
Lemma $\mathcal{O}(k)$ is a line bundle on $\mathbb{P}^n$ for all $k \in \mathbb{Z}$.

Pf We need to show local triviality

Pf We need to check local triviality.

$U_i = \{x_i \neq 0\}$ $\qquad [x_0 : \dots : x_n]$ homogeneous coords on $\mathbb{P}^n$.

$v = \left(\frac{x_0}{x_i} : \dots : 1 : \dots : \frac{x_n}{x_i}\right)$ = basis in ℓ



Any vector in ℓ: $\left(t\frac{x_0}{x_i} : \dots : t : \dots : t\frac{x_n}{x_i}\right) = tv$ $\qquad t$ = coord on ℓ

f = homog. polynomial on ℓ of deg. $k \Rightarrow$

$$f(tv) = \alpha t^k \qquad k \geq 0$$

$$f(\underbrace{x_0, \dots, x_n}_{\text{also a vector on } \ell}) = f(x_i \cdot v) = \alpha x_i^k$$

Claim: $\pi^{-1}(U) \simeq U \times \mathbb{A}^1$ isomorphism.

$$(\alpha t^k, \ell) \longleftarrow (\ell, \alpha)$$

$$(f, \ell) \longmapsto \left(\ell, \frac{f(x_0, \dots, x_n)}{x_i^k}\right)$$

For $k \leq 0$, we use the following linear algebra fact.

Fact Suppose V = 1-dim vector space over $\mathbb{K}$.

Then there is a canonical isomorphism

$$\left\{\begin{matrix}\text{degree } k \text{ polynomials} \\ \text{on } V^*\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{degree } (-k) \text{ Laurent} \\ \text{polynomials on } V\end{matrix}\right\}$$

Pf Let $V_k = \left\{\begin{matrix}\text{deg } k \text{ Laurent homogen} \\ \text{polynom on } V\end{matrix}\right\}$ $\qquad k \in \mathbb{Z}$

We have a pairing $V_k \otimes V_{-k} \longrightarrow \mathbb{K}$

$(f, g) \longrightarrow f \cdot g$ ← deg 0 poly = constant.

$$(f, g) \longrightarrow f \cdot g$$ deg 0 poly = constant.

Nondegenerate $\Rightarrow V_{-k} \simeq V_k^*$.

In coords: $\bar{e}$ = basis vector in V, any other vector $t \cdot \bar{e}$

$$g \in V_{-k}, \quad g(t\bar{e}) = \beta t^{-k}$$

$\bar{e}^*$ = dual basis on V^*, define map $V_{-k} \longrightarrow \{\text{deg } k \text{ poly on } V^*\}$

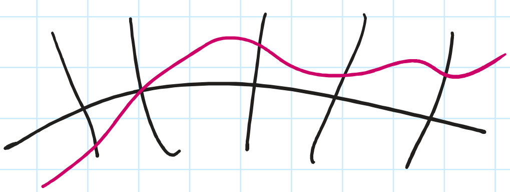
$\langle \bar{e}, \bar{e}^* \rangle = 1$ $\qquad \beta t^{-k} \longrightarrow h$ such that $h(s\bar{e}^*) = \beta s^k$.

Change basis: $\bar{e} \to \lambda \bar{e}$ $\quad t \to \frac{t}{\lambda}$ $\quad \bar{e}^* \to \frac{1}{\lambda}\bar{e}^*, s \to \lambda s$
coords

$$\beta t^{-k} = (\beta \lambda^{-k}) \cdot \left(\frac{t}{\lambda}\right)^{-k} \qquad \tilde{\beta} = \beta \lambda^{-k}$$

$$\beta s^k = (\beta \lambda^{-k}) \cdot (\lambda s)^k. \qquad \tilde{\beta} = \beta \lambda^{-k}$$

Def A section is a map $s: X \to E$ such that $\pi(s(x)) = x$ for all $x \in X$



Ex $E = X \times \mathbb{A}^1$ trivial bundle

Section $\longleftrightarrow$ function $X \to \mathbb{A}^1$.

Ex $E = \mathcal{O}(k)$ on $\mathbb{P}^n$, $k \geq 0$

F = degree k homog. polynomial on $\mathbb{A}^{n+1}$

$\rightsquigarrow$ section of $\mathcal{O}(k)$ $\qquad s(\ell) = (F|_\ell, \ell)$

So: degree k polynomials $\longrightarrow$ sections of $\mathcal{O}(k)$.

Lemma (a) s_1, s_2 = two sections of E

$\Rightarrow \frac{s_1}{s_2}$ = rational function on X

well defined on $\{s_2 \neq 0\}$.

(b) Assume E has a section which is nowhere zero.

then E is trivial.