Line bundles cont'd.

Recall $E \xrightarrow{\pi} X$ line bundle, $s: X \to E$ section if $\pi(s(x)) = x$.

Lemma (last time) (a) If $s(x) \neq 0$ for all x then E is trivial:

$$\begin{aligned} E &\xleftarrow{\sim} X \times \mathbb{A}^1 \\ t \cdot s(x) &\longleftarrow (x, t) \end{aligned}$$

(b) More generally, if $U = \{x : s(x) \neq 0\} \subset X$ open then E is trivial on U, that is, $\pi^{-1}(U) \simeq U \times \mathbb{A}^1$ by the same reason.

Notation: $\Gamma(E) = H^0(X, E)$ = space of global (that is, regular at all points of X) sections.

$X = \mathbb{P}^n$, $\quad \mathcal{O}(k) = \{(f, \ell) : \ell \in \mathbb{P}^n \leftrightarrow$ line in $\mathbb{A}^{n+1}$, f = degree k polynomial on $\ell\}$

Thm (a) $\mathcal{O}(k)$ is locally trivial

(b) $\Gamma(\mathcal{O}(k)) = \begin{cases} \text{degree } k \text{ homog. polynomials in } x_0, \ldots, x_n & k \geq 0 \\ 0, \text{ if } k < 0. \end{cases}$

Proof: (a) Consider the chart $U_i = \{x_i \neq 0\}$. On U_i, we have a section x_i^k of $\mathcal{O}(k)|_{\ldots}$

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$$x_i^k \neq 0 \Rightarrow \mathcal{O}(k)|_{U_i} \simeq U_i \times \mathbb{A}^1.$$

$v \longmapsto (x, \frac{v}{x_i^k})$ where $x = \pi(v)$.

(note: for $k<0$, x_i^k does not extend outside of U_i but it's OK)

(b) Suppose s = global section of $\mathcal{O}(k)$.

On U_i, we get a diagram:

$$\begin{tikzcd} \mathcal{O}(k)|_{U_i} \arrow[rr, "\sim", "{(x, \frac{v}{x_i^k})}"'] & & U_i \times \mathbb{A}^1 \\ U_i \arrow[u, "s"] \arrow[urr, "{x \mapsto (x, \frac{s(x)}{x_i^k})}"'] & & \end{tikzcd}$$

So $\frac{s(x)}{x_i^k}$ = regular function on U_i $= f_i(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i})$

$$s(x) = x_i^k \cdot f_i\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right) = x_i^k \cdot \frac{h_i(x_0, \dots, x_n)}{x_i^d}$$

where h_i = homog. of degree d, coprime to x_i

On $U_i \cap U_j$ we get

$$s(x) = x_i^{k-d} \cdot h_i(x_0, \dots, x_n) = x_j^{k-d'} \cdot h_j(x_0, \dots, x_n) \quad (*)$$

If $k<d$ then $x_i^{k-d} h_i(x_0, \dots, x_n)$ has denominator x_i^{d-k}

which contradicts $(*)$.

So $d \leq k$ and $s(x)$ is a homog. polynomial of degree k.

Conversely, any homog. polynomial of degree k in $x_0 \dots x_n$ restricts to a polynomial of deg. k on every line restricts to a polynomial of deg. k on every line

$\Rightarrow$ defines a global section of $\mathcal{O}(k)$. $\blacksquare$

Operations on line bundles

Lemma (a) E_1, E_2 = line bundles on $X \Rightarrow E_1 \otimes E_2$ line bundle

(b) E = line bdle $\Rightarrow E^*$ = line bdle. and $E \otimes E^*$ is trivial

(c) $f : X \longrightarrow Y$, $E \xrightarrow{\pi} Y$, E – line bdle on Y

Define $f^*E = \{(x, e) : x \in X, e \in E, f(x) = \pi(e)\}$

Then f^*E is a line bdle on X

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof: (a) $\dim V_1 = \dim V_2 = 1 \Rightarrow \dim V_1 \otimes V_2 = 1 \otimes 1 = 1$

Need to check local triviality: suppose

E_1 is trivialized on open $\mathcal{U}_\alpha$, so $\pi_1^{-1}(\mathcal{U}_\alpha) = \mathcal{U}_\alpha \times \mathbb{A}^1$

E_2 is trivialized on $\mathcal{U}'_\beta$, so $\pi_2^{-1}(\mathcal{U}'_\beta) = \mathcal{U}'_\beta \times \mathbb{A}^1$.

Consider all possible intersections $\mathcal{U}_\alpha \cap \mathcal{U}'_\beta$, then

$$\underbrace{(\pi_1 \otimes \pi_2)}_{\text{tensor product of fibers}}{}^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}'_\beta) = (\mathcal{U}_\alpha \cap \mathcal{U}'_\beta) \times (\mathbb{A}^1 \otimes \mathbb{A}^1) =$$
$$= (\mathcal{U}_\alpha \cap \mathcal{U}'_\beta) \times \mathbb{A}^1.$$

(b) Similarly, if $\dim V = 1$ then $\dim V^* = 1$,

and if $\pi^{-1}(U) = U \times \mathbb{A}^1$, then $(\underbrace{\pi^*}_{\text{dual in each fiber}})^{-1}(U) = U \times (\mathbb{A}^1)^* = U \times \mathbb{A}^1$

Also, $E \otimes E^*$ has a canonical section (Identity matrix) dual in each fiber

$\Rightarrow E \otimes E^*$ is trivial !

(c) Define $f^*E = \{(x, e) : x \in X, e \in E, \pi(e) = f(x)\}$

$$\begin{array}{cc} \downarrow & (x,e) \\ X & \downarrow \\ & x. \end{array}$$

fiber of f^*E over x = fiber of E over $f(x) \Rightarrow$ 1-d vector space

$\pi^{-1}(U) \cong U \times \mathbb{A}^1$

U = open in Y containing $f(x) \Rightarrow f^{-1}(U)$ open in X

$(f^*\pi)^{-1}(f^{-1}(U)) \simeq f^{-1}(U) \times \mathbb{A}^1$ (check it!)

$\Rightarrow f^*E$ is locally trivial.

Remark (a) If s_1 = section of E_1, s_2 = section of E_2

$\Rightarrow s_1 \otimes s_2$ is a section of $E_1 \otimes E_2$

(b) If $s : Y \to E$ section of E, and $f : X \to Y$

then f^*s is a section of f^*E

$f^*s(x) = (x, s(f(x)))$.

Def Picard group of X = $\{$line bdles on X, $\otimes\}$
Pic X

Inverse = E^*.

Thm (a) $\mathcal{O}(k) \otimes \mathcal{O}(m) \simeq \mathcal{O}(k+m)$, $\mathcal{O}(k)^* = \mathcal{O}(-k)$

Thm (a) $\mathcal{O}(k)\otimes\mathcal{O}(m)\simeq\mathcal{O}(k+m)$, $\mathcal{O}(k)^* = \mathcal{O}(-k)$

(b) $\mathcal{O}(k)\not\simeq\mathcal{O}(m)$ if $k\neq m$

Pf: (a) Clear, need to check in fibers

$$\begin{pmatrix}\text{deg } k \text{ polynoms}\\ \text{on a line } \ell\end{pmatrix}\otimes\begin{pmatrix}\text{deg } m \text{ poly}\\ \text{on } \ell\end{pmatrix}\simeq\begin{pmatrix}\text{deg } k+m \text{ poly}\\ \text{on } \ell\end{pmatrix}$$

(b) If $k, m>0$ then $\Gamma(\mathcal{O}(k)) = \left\{\begin{matrix}\text{deg } k \text{ poly}\\ \text{in } x_0, \dots, x_n\end{matrix}\right\}$

$\dim\Gamma(\mathcal{O}(k)) = \binom{n+k}{k}$, if $k<m$ then

$\binom{n+k}{k} < \binom{n+m}{m} \Rightarrow \mathcal{O}(k)\not\simeq\mathcal{O}(m)$.

If $k\geq 0$, $m<0$ then $\dim\Gamma(\mathcal{O}(k))>0$, $\dim\Gamma(\mathcal{O}(m))=0$

$\Rightarrow \mathcal{O}(k)\not\simeq\mathcal{O}(m)$

Finally, if $k, m<0$ then $\mathcal{O}(k)^* = \mathcal{O}(-k)$, $\mathcal{O}(m)^* = \mathcal{O}(-m)$

Since $\mathcal{O}(-k)\not\simeq\mathcal{O}(-m)$, we get $\mathcal{O}(k)\not\simeq\mathcal{O}(m)$. ∎

Fact (proof postponed) Any line bdle on $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(k)$ for some k $\Rightarrow$ $\operatorname{Pic}\mathbb{P}^n = \mathbb{Z}$.
